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THE

ELEMENTS OF ALGEBRA:

FOR THE USE OF

SCHOOLS AND COLLEGES.

BY

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PART I.

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PREFACE.

In writing the following Treatise the author has had primarily in view the development in natural order of the general laws that govern the operations of real quantities in Algebra. The various quantitative symbols have accordingly been introduced at as early a stage as possible, so as to bring them at the same time under the general rules. In the remaining parts of the book the treatment of the subjects will be found to agree, in the main, with the works of standard authors. Instead, however, of placing at the end a set of general examples, as was originally intended, the author was induced, from a consideration of the simplicity and importance of the subject, to insert a chapter on the method of Determinants.

In commencing the study of such a treatise as the following, the pupil should of course know something of the subject. To supply this preliminary knowledge the author intends to prepare forthwith a small treatise for the use of beginners.

Should the present work be received with favour by teachers a second part will be brought out, in which the higher parts of the subject will be treated.

The author would feel obliged for any suggestions or corrections that may be communicated to him.

University College,
Toronto, June, 1873.

THE ELEMENTS OF ALGEBRA.

INTRODUCTION.

1. In the ordinary processes of Arithmetic the quantities involved are expressed by means of the nine digits and zero, and combinations of these; but the operations performed are not indicated by the results obtained, although they may be sometimes suggested by the arrangement of the various parts of the process. We can always tell, for instance, when the operation of Division has been performed, by the arrangement of its various parts, but the connection between the several figures of the quotient and the corresponding steps of the process can only be exhibited by a lengthy explanation in words.

In Algebra, on the other hand, quantities, and the operations on quantities, are denoted by means of symbols and signs, the use of which enables us to exhibit operations, either simple or complex, on any quantities whatsoever without the assistance of words.

The results of these operations on *general* symbols do not exhibit the individual values of the quantities which are the subject of investigation. They only indicate the operations which ought to be performed on the given quantities to obtain the value sought. If, however, the given quantities are expressed by numbers, the results of the Algebraical will correspond to those of the Arithmetical operations, which are thus included in the more general processes of Algebra.

2. Any object or result of an Algebraic operation is called an Algebraic *quantity* or *expression.*

3. There are two distinct and independent elements in Algebraic quantities, the *designative* and the *quantitative;* the former of which is denoted by means of signs and combinations of signs, and the latter by means of symbols and combinations of symbols.

4. The usual symbols are (1) figures, as in Arithmetic; (2) letters of alphabets, either alone, or marked with accents, dashes, suffixes, or other marks, as a, b, x, y, α, β, a', x_1, a_2, β'', &c.

5. Quantities are said to be *positive* or *negative* according as their designative elements are denoted by the signs + or −, or the equivalents of these written immediately before their quantitative symbols.

Thus $+3$, $+x$, $+y$ are positive quantities whose quantitative elements are denoted by 3, x, and y; -5, $-a$, $-z$ are negative quantities whose quantitative values are 5, a, and z.

6. The sign + is called the *plus sign,* and the sign − the *minus sign.*

7. Positive and negative quantities are also called *real* quantities.

8. Quantities which are not real are called *imaginary*, or *impossible* quantities. The manner of denoting such quantities will only be treated of incidentally in the following Treatise. Unless otherwise stated, therefore, all quantities will be supposed real.

9. The symbol = stands for *is*, or *are*, *equal to*, and is written between the quantities whose equality it is desired to express. Thus $a = 3$ denotes that the value of a is 3.

10. The symbol $>$ stands for *is*, or *are*, *greater than*, and the symbol $<$ for *is*, or *are*, *less than*.

11. The symbol $\therefore$ means *hence*, or *therefore*, and the symbol $\because$ *since* or *because*.

12. The following are some of the laws by which symbols are combined to denote the quantitative element of Algebraic quantities :

(1.) The product of any quantitative symbols is represented by writing them down in a horizontal line one after another, in any order, with or without the multiplication sign ×, or the dot . , between them.

Thus ab, ba, $a.b$, $b.a$, $a \times b$, $b \times a$ all denote the product of a and b; abc, $a.b.c$, $a \times b \times c$, the product of a, b, and c; and so on.

Figure symbols are written first in order; thus $3a$, $5ab$, $6abc$.

When there are more figure symbols than one the multiplication sign × only must be used between them.

Thus the product of 5 and $6a$ is denoted by $5 \times 6a$, and not by $5.6a$, or $56a$, whose values are *five decimal six times a*, and *fifty-six times a*, respectively.

(2.) The symbol a^2 stands for aa, a^3 for aaa and generally a^n for $aa \times$ with a written n times.

(3.) The quotient of one quantity a by another b is denoted by the symbol $a \div b$, or the fractional form $\frac{a}{b}$.

(4.) The product of a fraction $\frac{a}{b}$ by a quantity c is denoted by $\frac{ac}{b}$, and the product of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is denoted by $\frac{ac}{bd}$.

Thus the product of $\frac{2}{a}$ and 3 is $\frac{6}{a}$; of $\frac{2}{3}$ and $\frac{a}{b}$ is $\frac{2a}{3b}$; and of $\frac{a}{3}$ and $\frac{2}{x}$ is $\frac{2a}{3x}$.

(5.) The quotient of a fraction $\frac{a}{b}$ by a quantity c is denoted by $\frac{a}{bc}$; of a quantity a by a fraction $\frac{b}{c}$ by $\frac{ac}{b}$; and of a fraction $\frac{a}{b}$ by a fraction $\frac{c}{d}$ by $\frac{ad}{bc}$.

Thus $\frac{3}{5} \div a = \frac{3}{5a}$; $3 \div \frac{5}{a} = \frac{3a}{5}$; $\frac{2}{a} \div \frac{5}{b} = \frac{2b}{5a}$.

13. It is to be observed that the numerical value of a quantity which is not fractional in form may be a fraction, and the numerical value of a quantity which is fractional in form may be an integer.

Thus a may stand for 2 and b for $\frac{1}{2}$, in which case the value of $\frac{a}{b}$ would be $2 \div \frac{1}{2}$ or 4, an integer.

Again, if $a = \frac{1}{2}$ and $b = \frac{1}{4}$, the value of ab would be $\frac{1}{2} \times \frac{1}{4}$, or $\frac{1}{8}$, and the value of $\frac{a}{b}$ would be $\frac{1}{2} \div \frac{1}{4}$, or 2.

14. When two or more quantities are multiplied together, each is said to be a *factor* of the product.

Thus a and b are factors of ab; 3 and a^2 are factors of $3a^2$; and 5, b and c are factors of $5bc$.

15. One factor of a product is said to be a *coefficient* of the remaining factor, and is said to be a *literal*, or a *numerical* coefficient according as it involves letters or not.

Thus in $3x$ the numerical coefficient of x is 3; in ax^2 the literal coefficient of x^2 is a.

Also, since $x = 1 \times x$, the coefficient of x in the quantity x is 1.

The sign + or − preceding a product is also a sign of the coefficient. Thus the coefficient of x in $+3x$ is $+3$, of x^2 in $-5ax^2$ is $-5a$, and of z^2 in $-\frac{2}{3}z^2$ is $-\frac{2}{3}$.

16. Quantities are equal when their designative elements are alike and their quantitative elements equal.

17. Quantities are said to be *like* or *unlike* according as their quantitative elements involve the same or different combinations of letters.

Thus $+5a$, $+7a$ are like quantities; and so also are $+6x^2y$, $-5x^2y$; $-2a$, $-a^2$ are unlike quantities.

ADDITION AND SUBTRACTION.

18. The *sum* of any quantities is denoted by writing them down in a horizontal line, one after another, in

any order; and each quantity is said to be a *term of* the sum.

Thus the sum of $+a$ and $-b$ is $+a-b$, or $-b+a$; the sum of $+a$, $-b$ and $+c$ is $+a-b+c$; and the sum of $+a-b+c$ and $+d-e$ is $+a-b+c+d-e$.

Also the terms of $+x-y$ are $+x$ and $-y$, and the terms of $+a^2-bc+c^2$ are $+a^2$, $-bc$, and $+c^2$.

19. If a quantity contains no parts connected by the signs $+$ or $-$ it is called a *simple* expression, or a *mononomial.*

Thus $2x$, $-3y$, $4ab$ are mononomials.

20. When a quantity consists of two terms it is called a *binomial* expression; when it consists of three terms it is called a *trinomial* expression; and generally when it consists of several terms it is called a *polynomial* or *multinomial* expression.

Thus $+2x-z$ is a binomial, $+a-b+c$ is a trinomial, and $+a-b-c+d$ a polynomial expression.

21. The sign of a mononomial, or of the first term of a polynomial, if it is positive, is generally omitted.

Thus $2x^2$ stands for $+2x^2$, and $a-b+c$ for $+a-b+c$

22. The sum of two quantities, positive and negative, whose quantitative elements are equal is 0.

Thus $+a-a=0$; $-2x^2+2x^2=0$; $+2a-3b-2a+3b$ $=+2a-2a-3b+3b=0$.

Examples.

1. The sum of $x-y$, $y-z$, and $z-x$ is

$$\begin{aligned} &x-y+y-z+z-x \\ =&x-x-y+y-z+z \\ =&0. \end{aligned}$$

2. The sum of $2x^2+x+1$, x^2-1, and $-2x^2-1$ is $2x^2+x+1+x^2-1-2x^2-1$, which becomes by changing the order of the terms

$$2x^2-2x^2+x+1-1+x^2-1$$
$$=x+x^2-1.$$

Exercises, I.

Find the sum of

1. $a+b-c$ and $-b+c$.
2. $a+b$, $a-b+c$, and $b-c-a$.
3. $a-b+c$, $b-c+a$, and $c-a+b$.
4. $\frac{1}{2}xy-\frac{1}{3}yz+\frac{1}{4}zx$, $\frac{1}{3}yz-\frac{1}{2}xy$, and $xy-\frac{1}{4}zx$.

23. The *difference* between one quantity and another is the quantity which added to the latter will produce the former.

Thus the difference between $+a$ and $-b$ is the quantity which added to $-b$ will produce $+a$.

24. The operation of Subtraction is thus the inverse of Addition, and therefore the difference required will be found by adding the first quantity to the second with its sign or signs changed.

Thus the difference between a and $-b$ is $a+b$, because the sum of $a+b$ and $-b$ is a. So also the difference between $a-b$ and $c-d$ is the sum of $a-b$ and $-c+d$, that is $a-b-c+d$, because the sum of $a-b-c+d$ and $c-d$ is $a-b$.

Example.

From $a+b-c$ take $b-c+d$.

Change the sign of every term in $b-c+d$, and we get $-b+c-d$.

Therefore the difference required is

$$a+b-c-b+c-d=a-d.$$

EXERCISES, II.

1. From $a-b$ take $-b+c$.
2. From $2x-3y$ take $2x-3y+z$.
3. From $a+b-c$ take $b-c+d$.
4. From $a-b+c+d$ take $-b+d-e$.

25. The results of operations in Addition and Subtraction may also be represented by combinations of signs and symbols, according to the following Laws: Law I.

$$+(+a)=+a,\ +(-a)=-a,\ -(+a)=-a,$$
$$-(-a)=+a.$$

Thus the sum of a and $+b$, which is $a+b$, may also be written $a+(+b)$ if $+(+b)=+b$; the sum of a and $-b$, which is $a-b$, may be written $a+(-b)$, if $+(-b)=-b$; the difference between a and $+b$, which is $a-b$, may be written $a-(+b)$, if $-(+b)=-b$; and the difference between a and $-b$, which is $a+b$, may be written $a-(-b)$, if $-(-b)=+b$.

26. The bracket ( ) is usually omitted in the case of a mononomial.

Thus $+\ (-a)$ is written $+-a$.

27. Law II.

$$+(+a+b)=++a++b=+a+b,$$
$$+(+a-b)=++a+-b=+a-b,$$
$$-(+a+b)=-+a-+b=-a-b,$$
$$-(+a-b)=-+a--b=-a+b,$$

and so on, the sign outside the bracket being said to *operate distributively* on all the signs within.

Thus the sum of a and $-b+c$, which is $a-b+c$, may also be written $a+(-b+c)$ if $+(-b+c)=-b+c$; and the difference between a and $+b-c$, which is $a-b+c$, may be written $a-(+b-c)$ if $-(+b-c)=-b+c$.

So also the sum and difference of a and $-b+c-d$ may be written $a+(-b+c-d)$, and $a-(-b+c-d)$, respectively.

28. In the above combinations signs other than those written immediately before symbols are called signs of *operation*, the sign + being the sign of operation in the case of Addition, and the sign − in the case of Subtraction; and all those quantities whose terms are affected by such a sign are enclosed within brackets ( ), { }, [ ].

29. The bracket is sometimes replaced by a *vinculum* ——— drawn above the whole quantity.

Thus $a-\overline{+b-c}$ is equivalent to $a-(+b-c)$.

30. Also the sign of the first term of the quantity within the bracket or under the vinculum, if it is positive, is generally omitted.

Thus $a-(b-c)=a-(+b-c)$.

EXAMPLES.

1. Enclose within a bracket preceded by + the second and third terms of $2a-3b+c$.

Here $2a-3b+c=2a+-3b++c=2a+(-3b+c)$.

2. Enclose within a bracket preceded by − the second and third terms of $2x-y+z$.

Here $2x-y+z=2x-+y--z=2x-(+y-z)=2x-(y-z)$.

EXERCISES, III.

1. Enclose within a bracket preceded (1) by +, (2) by −, the second and third terms of

$$a+b+c,$$
$$a+b-c,$$
$$a-b+c,$$
$$a-b-c,$$
$$a-b+c-d.$$

2. Enclose within brackets preceded by $-$ the first and second, and third and fourth terms of

$$x-y+z-1,$$
$$-x-y+z-1,$$
$$-x+y+z+1,$$
$$x+y-z-1.$$

31. Law III.

$$+ab=+a(+b)=-a(-b),$$
$$-ab=+a(-b)=-a(+b),$$
$$+(+ab+ac)=+a(+b+c),$$
$$+(-ab+ac)=+a(-b+c),$$
$$-(+ab-ac)=-a(+b-c),$$
$$+(ab-ac-ad)=+a(b-c-d),$$

and so on, the symbol outside the bracket being said to *operate* as a multiplier on each symbol within.

Thus $5x-5y=5(x-y)$; $a-3b-3c=a-(3b+3c)$ $=a-3(b+c)$; $a-\frac{1}{2}b+\frac{1}{2}c=a-(\frac{1}{2}b-\frac{1}{2}c)=a-\frac{1}{2}(b-c)$; $2x-a^2+ab=2x-(a^2-ab)=2x-a(a-b)$.

Exercises, IV.

Enclose the second and third terms of the following quantities within a bracket preceded by the symbol common to those terms:

1. $2a-xy-xz$.
2. a^2-b^2+bc.
3. $a-2bc+2c-d$.
4. $x-\frac{1}{2}y-\frac{1}{2}z$.
5. $2a-\frac{2}{3}pq+\frac{2}{3}p^2$.

32. Law IV.

$$1+1=2,\ 2+1=3,\ 3+1=4,$$

and so on.

33. From this and the preceding Laws it follows that when the symbols are figures, Algebraical Addition includes Arithmetical Addition and Subtraction of whole numbers and fractions if quantities to be added are denoted by $+$, and quantities to be subtracted by $-$.

Thus the Arithmetical operations

$$\begin{array}{ccc} & & 7\\ 5 & & 8\\ \cline{3-3} 2 & 8 & 15\\ 4 & 3 & 11\\ \hline 11 & 5 & 4\end{array}$$

may be denoted by the Algebraical operations

$$5+2+4=11,\ 8-3=5,\ 7+8-11=4.$$

So also $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{1}{2}(1+1+1)=+\frac{3}{2}$, $\frac{3}{4}-\frac{2}{3}=\frac{9}{12}-\frac{8}{12}$ $=(9-8)\frac{1}{12}=\frac{1}{12}$.

The following examples, however, of Algebraical Addition have no corresponding place in Arithmetic:—

$$-1-2-3=-(1+2+3)=-+6=-6,$$
$$+4-5=+4-4-1=-1,$$
$$\tfrac{4}{5}-\tfrac{5}{6}=\tfrac{24}{30}-\tfrac{25}{30}=(24-25)\tfrac{1}{30}=-\tfrac{1}{30}$$

34. From the preceding Laws it follows that

$$a+a=(1+1)\,a=2a,$$
$$-a-a-a=-(1+1+1)\,a=-3a,$$
$$7a-5a=(7-5)\,a=2a,$$
$$\tfrac{1}{4}a+\tfrac{1}{4}a+\tfrac{1}{4}a=\tfrac{1}{4}(1+1+1)\,a=\tfrac{3}{4}a,$$
$$\tfrac{2}{3}a-\tfrac{3}{4}a=\tfrac{8}{12}a-\tfrac{9}{12}a=(8-9)\,\tfrac{1}{12}a=-\tfrac{1}{12}a,$$

and so on.

35. Hence the sum of any number of like quantities is a like quantity whose coefficient is the Arithmetical difference between the sums of the positive and negative coefficients, and has the sign of the numerically greater sum.

If all the coefficients have the same sign the sum will have that sign, and be equal numerically to their sum.

36. The operations of Addition and Subtraction may now be conducted by arranging the expressions

under each other so that like terms shall stand in the same column, and proceeding as in the following

EXAMPLES.

$$\begin{array}{r} 2a+3b \\ a-\;\;b \\ \hline 3a+2b \end{array} \qquad \begin{array}{r} 2x^2-3x+1 \\ x^2+3x+2 \\ \hline 3x^2 \qquad +3 \end{array} \qquad \begin{array}{r} a+b-\;\;c \\ 2a-b+2c \\ a-b+3c \\ \hline 4a-b+4c \end{array}$$

In the first example the coefficients in the first column are 2 and 1, and as they have the same sign their sum is 3; in the second column the coefficients are $+3$ and -1, and as they have different signs their numerical difference 2 is taken preceded by $+$, the sign of the numerically greater coefficient 3.

$$\begin{array}{r} 2ax^2-3bx+c \\ -3ax^2-\;\;bx+2c \\ 2ax^2+2bx-4c \\ \hline ax^2-2bx-c \end{array} \qquad \begin{array}{lllll} 7x^2 & -3xy & & -x & \\ 3x^2 & & -y^2 & +3x & \\ -2x^2 & +4xy & +5y^2 & & -y \\ & -7xy & -\;\;y^2 & +9x & \\ \hline 8x^2 & -6xy & +3y^2 & +11x & -y \end{array}$$

$$\begin{array}{r} \frac{3}{4}x^2-2x+\frac{1}{3} \\ -\;\;x^2+x\;\;-2 \\ 2x^2-x\;\;+\frac{1}{2} \\ \hline \frac{7}{4}x^2-2x-\frac{7}{6} \end{array}$$

In the first column of the last Example the sum of the positive coefficients is $2\frac{3}{4}$, and the numerical difference between this and the negative coefficient 1 is $\frac{7}{4}$, to which the sign of $2\frac{3}{4}$ is prefixed.

EXERCISES, V.

Add together

1. $5a^3$, $6a^3$, $7a^3$.
2. $-7a$, $-3a$, $-a$.
3. $2x$, $-3x$, $7x$.
4. $3x+2y$, $5x-6y$, $-7x+8y$.
5. $-6x+3y$, $-2x-8y$, $8x+5y$.

6. $8x^2-17x+12,\ -x^2+3x-9,\ -7x^2+10x-2,\ x^2-x-1.$

7. $\frac{1}{2}a+b,\ \frac{1}{2}a-b.$

8. $\frac{3}{2}a+b-\frac{1}{2}c,\ -\frac{1}{2}a-\frac{1}{3}b+\frac{1}{4}c.$

9. $x^3-2ax^2+a^2x+a^3,\ x^3+3ax^2,\ 2a^3-ax^2-2x^3.$

10. $2ab-3ax^2+2a^2x,\ 12ab+10ax^2-6a^2x,\ -8ab+ax^2-5a^2x.$

11. $\frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{4}y^2,\ -\frac{1}{3}x^2+\frac{1}{4}xy-\frac{1}{5}y^2.$

12. $\frac{5}{2}x-\frac{7}{3}y,\ -\frac{4}{3}x+\frac{5}{2}y,\ \frac{7}{4}x-\frac{3}{2}y.$

13. $a^2-b^2+\frac{1}{6}c^2,\ a^2+b^2-\frac{1}{3}c^2,\ b^2-a^2+\frac{1}{6}c^2.$

14. $\frac{a}{2}-\frac{b}{3}+\frac{c}{4},\ \frac{b}{2}-\frac{c}{3}+\frac{a}{4},\ \frac{c}{2}-\frac{a}{3}+\frac{b}{4}.$

37. In Art. 24 it was shown that the difference between one quantity and another was the sum of the former, and the latter with its sign or signs changed. Hence the process of Subtraction can be conducted as in the following

EXAMPLES.

1. From $7a+14b$ take $5a-6b$.

Here the second quantity with its signs changed is $-5a+6b$, which is to be added to $7a+14b$.

$$\begin{array}{r} 7a+14b \\ -5a+6b \\ \hline 2a+20b \end{array}$$

It is usual, however, to put down the result without actually changing the sign, but merely supposing it done, as in the following examples:—

2. From $3x-2y+z$ take $2x-5y+3z$.

$$\begin{array}{r} 3x-2y+z \\ 2x-5y+3z \\ \hline x+3y-2z \end{array}$$

Here the first term of the difference is $3x-2x=x$, the second term is $-2y+5y=+3y$, and the third term is $+z-3z=-2z$.

3. From $a^3+3a^2b+3ab^2+b^3$ subtract $a^3-3a^2b+3ab^2-b^3$.

$$\begin{array}{r}
a^3+3a^2b+3ab^2+b^3 \\
a^3-3a^2b+3ab^2-\ b^3 \\
\hline
+6a^2b \qquad\quad +2b^3
\end{array}$$

4. From $2x^2-x+5$ subtract $-x^3+x^2+3$.

$$\begin{array}{r}
2x^2-x+5 \\
-x^3+x^2 \qquad +3 \\
\hline
x^3+x^2-x+2
\end{array}$$

5. From $\frac{1}{2}a-b+\frac{1}{3}c$ subtract $a+\frac{1}{2}b+c$

$$\begin{array}{r}
\frac{1}{2}a-b\ +\frac{1}{3}c \\
a+\frac{1}{2}b+c \\
\hline
-\frac{1}{2}a-\frac{3}{2}b-\frac{2}{3}c
\end{array}$$

EXERCISES, VI.

1. From $2a-5b$ subtract $a+2b$.

2. From $3x-7y+4z$ subtract $2x-3y+2z$.

3. From $x-y+z$ subtract $-x-y-z$

4. From $-2a-3b+2c$ subtract $2a+b-2c$.

5. From $4x^4-3x^3-2x^2-7x+9$ subtract $x^4-2x^3-2x^2+7x-9$.

6. From $-5a^2+3b^2+c^2-7ab+8bc-3ca$ subtract $-7a^2-3b^2+6c^2+9ab-4bc-5ca$.

7. From $\frac{1}{2}a+b$ subtract $\frac{1}{2}a-b$.

8. From $a+b$ subtract $\frac{1}{2}a-\frac{1}{2}b$.

9. From $\frac{1}{2}x^2-\frac{5}{6}x+\frac{1}{3}$ subtract $-\frac{1}{2}x^2+\frac{1}{6}x-\frac{2}{3}$.

10. From $\frac{1}{2}yz-\frac{1}{3}zx+\frac{1}{4}xy$ subtract $\frac{1}{3}yz-\frac{1}{4}zx-\frac{1}{5}xy$.

THE SIGNS + AND −.

38. In the preceding Articles it will be observed that the signs + and − are employed to denote (1) the designative elements of quantities, (2) the operations of Addition and Subtraction. They are also used in combination in a twofold sense to denote the operations of Addition and Subtraction of quantities, and the designative elements of the quantities on which those operations are to be performed. These various uses of the signs will be illustrated in the following Articles:—

39. Let the sign + denote that a quantity is *additive*, or *to be added;* then the sign − will denote that a quantity is *subtractive*, or *to be subtracted.*

Thus if 40 and 50 are to be added to 100 the 40 and 50 and the operation of Addition may be denoted by $+40$, $+50$ the sum of which is $+90$. That is, the result will be the same if 90 be added to 100.

If 40 is to be subtracted from, and 50 added to 100, the 40 and 50 and the operations of Subtraction and Addition may be be denoted by -40, $+50$, the Algebraic sum of which is $+10$. That is, the result will be the same if 10 be added to 100.

If 40 is to be added to and 50 subtracted from 100, the 40 and 50 and the operations of Addition and Subtraction may be denoted by $+40$, -50, the Algebraic sum of which is -10. That is, the result will be the same if 10 be subtracted from 100.

40. Again, suppose a man to walk along a straight road (running east and west) from A to B and then to C, so that AC is his distance from the starting-point A.

Here if the + sign denote *east* the – sign will denote *west*.

If he walk from A to B east 10 miles, this distance and direction may be denoted by $+10$. If now he walk from B to C east 4 miles, this distance and direction may be denoted by $+4$; and the distance AC will be denoted by $+10+4$ or $+14$; that is, C will be 14 miles east of A.

If AB be 10 miles east, and BC 4 miles west, BC will be denoted by -4, and AC will be $+10-4$ or $+6$; that is, C will be 6 miles east of A.

If AB be 10 miles east, and BC 15 miles west, BC will be denoted by -15, and AC by $+10-15$ or -5; that is, C will be 5 miles west of A.

41. If + written before a sum denote that it is *due to* a person, then – will denote that it is *due by* him.

Thus if 60 dollars be due to and 50 dollars due by him, the former may be denoted by $+60$ and the latter by -50, and the Algebraic sum of these, which is $+10$, will denote that 10 dollars are due *to* him.

Again, if 50 dollars be due to, and 60 dollars due by him, the Algebraic sum of these, which is -10, will denote that 10 dollars are due *by* him.

42. The meanings of the signs + and – in combinations may be illustrated by the following case:—

If a man's *assets* be denoted by + his *liabilities* will be denoted by –, and + and – when written before the former signs will denote the operations of Addition and Subtraction.

Thus $++$ will denote the Addition of an asset, $+-$ the Addition of a liability, $-+$ the Subtraction of an asset, and $--$ the Subtraction of a liability.

For example, suppose a man's assets to be 1000 dollars, and his liabilities 400 dollars; the former may be denoted by $+1000$, and the latter by -400, and the man's worth will be the sum of these, that is,

$$+1000+-400=+600=+1000-+400.$$

Therefore the addition of a liability of 400 is the same as the subtraction of an asset of 400.

Again, suppose his assets to be 1,000 dollars, and his liabilities 300, so that he is worth $1{,}000+-300$, or 700 dollars. If, however, his liability of 300 be subtracted his worth will be 1,000; that is, his worth will be $+1{,}000 + -300 - -300 = +1{,}000$.

Therefore the subtraction of a liability of 300 is equivalent to the addition of an asset of 300.

43. Hitherto letter symbols have been employed to denote the quantitative elements only of quantities, but they may now be employed to denote both the designative and quantitative elements. In such cases the signs + and − connecting them are signs of operation.

Thus the sum of $+a$ and $-b$ which is $+a+-b$ may be represented by $x+y$, if $x=+a$, $y=-b$. Here the + in $x+y$ denotes the operation of Addition, the symbol x denotes a positive quantity and the symbol y a negative quantity.

Again, the difference between $+a$ and $-b+c$ which is $+a-(-b+c)$, or $+a - -b - +c$, may be represented by $x-y-z$, if $x=+a$, $y=-b$, $z=+c$.

44. It appears, therefore, that the sign of such a quantity as $a+b$ cannot be determined until we know the magnitudes and signs of a and b.

Thus if a be a positive and b a negative quantity, $a+b$ will be positive if a be numerically greater than b, and negative if a be numerically less than b. For example, if $a=+6$, $b=-4$, $a+b=+6+-4=+2$, a positive quantity; but if $a=+6$, $b=-8$, $a+b=+6+-8=-2$, a negative quantity. So if a be positive and b be negative, the quantity $a-b$ will be positive.

For example, if $a = +7$, $b = -5$, $a-b = +7 - -5 = +12$, a positive quantity.

45. One quantity a is said to be Algebraically greater or less than another b, according as their difference $a-b$ is positive or negative.

Thus, $+8 > +7$, $+5 > -6$, $-2, > -6$, $0 > -4$, because the difference between the quantities is in each case positive.

Again, $+5 < +7$, $-8 < -3$, $-2 < 0$, because the differences are negative, being -2, -5, -2, respectively.

Hence it follows that such a series of quantities as

$$-4, -2, -1, 0, 1, 2, 4,$$

is in ascending order of magnitude, whilst the quantities

$$5, 3, 2, 0, -1, -4, -7,$$

are arranged in descending order of magnitude.

BRACKETS.

46. The use made of brackets in Art. 27 may be extended, so as to enable us to collect within a bracket various parts of an expansion in which a bracket is already employed. When it is necessary to use more brackets than one they are generally made of different shapes.

The following examples will illustrate the mode of introducing additional brackets.

EXAMPLES.

1. Since the sum of $-b$ and $-c+d = -b+(-c+d)$ the sum of a and $-b+(-c+d)$ may be written

$$a+\{-b+(-c+d)\}.$$

So also the difference between a and $-b+(-c+d)$ may be written

$$a-\{-b+(-c+d)\}.$$

2. The quantity $a-b+c+d-e$ may be written in the equivalent forms

$$a-\{+b-c-d+e\}$$
$$a-\{+b-(c+d-e)\}$$
$$a-\{+b-(c-\overline{-d+e})\}.$$

Here the signs of the terms within the bracket first introduced must be changed on account of the sign $-\{$.

In the same manner the signs of the terms within ( ) are changed on account of the sign $-($; and the signs of the terms under the vinculum on account of the $-$ sign preceding it.

3. The quantity $a-b+c+d-e$ may also be expressed in the equivalent forms

$$a+\{-b+c+d-e\}$$
$$a+\{-(b-c)-(-d+e)\}.$$

Here the introduction of the bracket preceded by $+$ does not affect the signs.

Exercises, VII.

Enclose within a bracket preceded by $-$ all but the first term of

1. $a-b+(c-d)$
2. $a+b-(c-d)$
3. $a-b-(c+d)$
4. $a-(b-c)+d$
5. $a+(b-c)-d$
6. $a-(b-c)-d$
7. $a-b-\{-c+(d-e)\}$.

77. Conversely, an expression may be freed from brackets *beginning with the inside pair* by removing

them in succession, the signs of the terms within any bracket being retained, or changed, according as the sign immediately preceding is + or −.

EXAMPLES.

1. $$\begin{aligned} a-\{-b-(c-d)\} &= a-\{-b-c+d\} \\ &= a+b+c-d. \end{aligned}$$

2. $$\begin{aligned} 2a-3b-\{-3b+(2a-c)\} &= 2a-3b-\{-3b+2a-c\} \\ &= 2a-3b+3b-2a+c \\ &= c. \end{aligned}$$

3. $$\begin{aligned} &2x-[3y-\{-2z-(2x-\overline{2z+3y})\}] \\ &=2x-[3y-\{-2z-(2x-2z-3y)\}] \\ &=2x-[3y-\{-2z-2x+2z+3y\}] \\ &=2x-[3y+2z+2x-2z-3y] \\ &=2x-3y-2z-2x+2z+3y \\ &=0. \end{aligned}$$

48. If the brackets be removed successively, commencing with the outside ones, the sign immediately preceding any bracket will not affect the signs of quantities within the other brackets.

Thus the preceding example may be treated as follows:

$$\begin{aligned} &2x-[3y-\{-2z-(2x-\overline{2z+3y})\}]=2x-3y+\{-2z-(2x-\overline{2z+3y})\} \\ &=2x-3y-2z-(2x-\overline{2z+3y}) \\ &=2x-3y-2z-2x+\overline{2z+3y} \\ &=2x-3y-2z-2x+2z+3y \\ &=0. \end{aligned}$$

49. Expressions may also be freed from brackets by retaining the several combinations of signs till all the brackets are removed, and then replacing each combination by its equivalent + or − sign according to the law,

A combination of the signs $+$, $-$ is equivalent to $-$, or $+$, according as it contains an odd number of $-$ signs or not.

For example,

$$+-+a=+-a=-a$$
$$--+a=--a=+a$$
$$+--+a=+--a=++a=+a.$$

Thus

$$\begin{aligned} a-\{-b+(-c-d)\} &= a-\{-b+-c+-d\} \\ &= a--b-+-c-+-d \\ &= a+b+c+d. \end{aligned}$$

50. Hence the brackets may be removed simultaneously by the law that each sign of operation affects all succeeding terms as far as its accompanying bracket extends.

For example, $a-[-b-(+c-\overline{d+e})]$

$$\begin{aligned} &= a- -b - - +c - - -d - - - +e \\ &= a+b+c-d-e. \end{aligned}$$

Exercises, VIII.

Simplify the following expressions by removing the brackets and collecting like terms:

1. $x-(-y-z)-(z+x)$
2. $(a-b)-(3b-1)$
3. $(a+x)-(b-x)-(a-b)$
4. $2x-\{-3y-(-2x+4z)\}$
5. $\{a-(x-y)\}-(2a+x+y)$
6. $3a-\{b+(2a-b)-(a-b)\}$
7. $2a-b-\{-(c-d)-(-2a+b+d)\}$
8. $6a-[4b-\{-4a-(6a-4b)\}]$
9. $16-\{5-2x-[1-(3-x)]\}$
10. $a+b-[-c+d-\{-a-(b-\overline{d-e})\}]$
11. $[a-5b-\{a-(5c-\overline{2c-b}-4b)+2a-(a-\overline{2b+c})\}]$.

51. In the same manner as the use made of brackets in Art. 27 was extended in the preceding Articles

may the use made of brackets in Art. 31 be extended.

For example, the quantity $ab-acd+ace$ may be written in the equivalent forms

$$a\,\{b-cd+ce\},$$
$$a\,\{b-c\,(d-e)\}.$$

So also the quantity $3-3x-3xy+3xyz$ may be written in the equivalent forms

$$3\;[1-x-xy+xyz]$$
$$3\;[1-x\{1+y-yz\}]$$
$$3\;[1-x\{1+y(1-z)\}].$$

52. In such cases, the effect of each symbol of operation extends as a multiplier of the terms after it as far as its accompanying bracket.

Thus in the last example 3 is a multiplier of every term as far as] .

53. Conversely, such quantities as the last may be freed from brackets by the law, that each sign and symbol of operation affect all terms as far as their accompanying bracket extends.

Thus if the brackets be removed successively, commencing with the inside ones,

$$\begin{aligned}4\,[1-2x\{-1-2y(1-z)\}] &= 4\;[1-2x\{-1-2y+2yz\}]\\ &= 4\;[1+2x+4xy-4xyz]\\ &= 4+8x+16xy-16xyz.\end{aligned}$$

The brackets may also be removed simultaneously, as in the following example :

$$\begin{aligned}&2\;[1-x\,\{-1+2y\,(-1+z)\}]\\ &\qquad = 2\;-\;-2x-+-2x\times 2y-++2x\times 2yz\\ &\qquad = 2\;+2x+4xy-4xyz.\end{aligned}$$

EXERCISES, IX.

Simplify the expressions

1 $(a+b)x+(a-b)y$.
2. $a-2(3a+b)+3(2a-b)$.
3. $3[a-2\{b-(c+d)\}]$.
4. $3a-[2a-2\{a-(a-1)\}+2]$.
5. $a-2(3a+b)-3\{b+2(a-b)\}$.
6. $4a-[2a-\{2b(x+y)-2b(x-y)\}]$.
7. $a-2[b+3\{a-2(b-c)+2b-3(a-b+2c)\}]$.
8. $a-3(b-c)-\frac{1}{2}\{(a-b)-4(\overline{b-c}-\frac{1}{3}\overline{a-b})\}$.

54. The use of brackets may be still further extended to express such a quantity as $a(c+d)+b(c+d)$ in the equivalent form $(c+d)(a+b)$.

In the same manner

$$a(-c+d)-b(-c+d)=(-c+d)(a-b).$$
$$(a-b)x-(a-b)y-(a-b)z=(a-b)(x-y-z).$$
$$\begin{aligned} xy-3x+2y-6&=x(y-3)+2(y-3).\\ &=(y-3)(x+2).\end{aligned}$$
$$\begin{aligned}(a+1)x+(a+1)y-a-1&=(a+1)x+(a+1)y-(a+1)\\ &=(a+1)(x+y-1).\end{aligned}$$

MULTIPLICATION.

55. The *product* of any quantities is represented by enclosing each of them in a bracket, and writing them together in a horizontal line in any order with or without the dot ., or Multiplication sign × between them.

Thus $(+a)(-b)$ denotes the product of $+a$ and $-b$; $(a-b)(-c)$ denotes the product of $a-b$ and $-c$; $(a-b)(c-d)$ denotes the product of $a-b$ and $c-d$; and $(a-b)(c-d)(e-f)$ denotes the product of $a-b$, $c-d$, and $e-f$.

56. Each of the quantities so enclosed in a bracket is called a *factor* of the product.

Thus $+a$, $-b$, and $+c$ are factors of $(+a)(-b)(+c)$.

57. When a factor is a mononomial it is called a *simple* factor; otherwise, a *compound* factor.

Thus $-a$ is a simple factor, and $b-c$ a compound factor of $(-a)(b-c)$.

When the factor written first is a simple factor, its bracket is generally omitted.

Thus $(-a)(-b)=-a(-b)$, $(+x)(y-z)=+x(y-z)$.

58. From the preceding mode of representing a product, it appears that it is equivalent to the sum of certain terms which can be obtained by laws the converse of those laid down in previous Articles for expressing sums in various equivalent forms, Multiplication being thus another form of Algebraical Addition.

59. It is convenient to make three cases in Multiplication, namely,

I. The Multiplication of simple factors.

II. The Multiplication of a simple and a compound factor.

III. The Multiplication of compound factors.

We shall take these three cases in order.

60. I. The product of two simple factors is obtained by a law the converse of that given in Art. 31, namely,

$$\begin{aligned} +a(+b) &= + +ab = +ab, \\ +a(-b) &= + -ab = -ab, \\ -a(+b) &= - +ab = -ab, \\ -a(-b) &= - -ab = +ab. \end{aligned}$$

Here it will be observed that the sign of the product may always be obtained from the signs of its two factors by the law that *like signs produce* +, *and unlike signs* −.

Thus the product of $-2x$ and $-3y$ is $+6xy$, the product of $+4x^2$ and $-3y$ is $-12x^2y$.

61. Again, since $+a(-b) = -ab$, it follows that

$$+a(-b)(+c) = -ab(+c) = -abc.$$

In the same manner it may be shown that

$$-a(+b)(-c)(+d) = +abcd,$$

and generally that the product of any number of factors is obtained by finding the product of the first and second, and multiplying this result by the third, and so on.

62. In addition to the modes given in Art. 12, for representing the quantitative elements of quantities we shall now use the symbol a^n, where n may be positive or negative, integral or fractional.

63. The symbol a^n is called the *nth power of a*, or *a to the power of n*, and this power is said to be of the *nth degree;* also the symbol n is called the *index*, or *exponent* of the power.

Thus a^2 is called a to the power of 2, or *a squared;* a^3 is called a to the power of 3, or *a cubed;* whilst a^4, a^{-3}, $a^{\frac{3}{4}}$, $a^{\frac{-5}{6}}$ are called a to the power of 4, a to the power of -3, a to the power of $\frac{3}{4}$, and a to the power of $-\frac{5}{6}$, respectively. Also the index of x^5 is 5, and of $x^{-\frac{2}{3}}$ is $-\frac{2}{3}$.

The meanings of such symbols will be explained farther on.

64. Powers, and the products of powers of a quantity are represented according to the following laws, called *Index Laws*.

$$\text{I. } a^m a^n = a^{m+n},$$
$$\text{II. } (a^m)^n = a^{mn},$$
$$\text{III. } (ab)^n = a^n b^n,$$
$$\text{IV. } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

m and n being any real quantities whatsoever, including zero.

Thus, by Law I. we have

$$a^5.a^6 = a^{5+6} = a^{11},$$
$$a^{-2}.a^4 = a^{-2+4} = a^2,$$
$$a^{\frac{1}{3}}.a^{\frac{1}{2}} = a^{\frac{1}{3}+\frac{1}{2}} = a^{\frac{5}{6}},$$
$$a^{-\frac{1}{2}}.a^{\frac{1}{3}} = a^{-\frac{1}{2}+\frac{1}{3}} = a^{-\frac{1}{6}}.$$

By Law II. it follows that

$$(a^2)^5 = a^{10},\ (a^{\frac{1}{2}})^3 = a^{\frac{3}{2}},$$
$$(a^2)^{\frac{1}{3}} = a^{\frac{2}{3}},\ (a^{-\frac{1}{3}})^{\frac{1}{2}} = a^{-\frac{1}{6}}.$$

By Law III.

$$(ab^2)^3 = a^3b^6,\ (a^{-1}b^3)^4 = a^{-4}b^{12},$$
$$(a^2b^{-1})^{\frac{1}{2}} = ab^{-\frac{1}{2}}.$$

65. The meaning of such symbols as $a^{\frac{1}{2}}$, $a^{\frac{2}{3}}$, $a^{\frac{5}{4}}$ can now be established. For by the second Index Law,

$$(a^{\frac{1}{2}}) = a^2,$$

or $a^{\frac{1}{2}}$ is a quantity whose square is a; that is, in Arithmetical language, $a^{\frac{1}{2}}$ is the square root of a, or

$$a^{\frac{1}{2}} = \sqrt{a}.$$

Again, $(a^{\frac{2}{3}})^3 = a^2$, or $a^{\frac{2}{3}}$ is a quantity whose cube is a^2; that is, $a^{\frac{2}{3}}$ is the cube root of a^2, or $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

In like manner, since $(a^{\frac{m}{n}})^n = a^m$, it may be shown that $a^{\frac{m}{n}} = \sqrt[n]{a^m}$; so that, m and n being positive integers, the symbols $a^{\frac{1}{n}}$, $a^{\frac{m}{n}}$ may be called in Arithmetical language the nth root of a, and the nth root of the mth power of a respectively.

Thus $2^{\frac{1}{3}}$ is called 2 to the power of $\frac{1}{3}$, or the cube root of 2; and $3^{\frac{4}{5}}$ is called 3 to the power of $\frac{4}{5}$, or the 5th root of the 4th power of 3.

66. By the first Index Law it follows that $a^3a^{-3} = a^{3-3} = a^0$. The meaning of such a result may be established as follows:—

$$a^m.a^0 = a^{m+0} = a^m.$$

Also $$a^m \times 1 = a^m.$$

Therefore $$a^0 = 1.$$

Thus $(x^2)^0 = x^0 = 1$; $(y^{\frac{1}{3}})^0 = y^0 = 1$; $2x^2.x^{-2} = 2x^0 = 2$, $4x^{\frac{1}{2}} \times 3x^{-\frac{1}{2}} = 12x^0 = 12$.

67. The meanings to be assigned in accordance with the Index Laws to such symbols as a^{-1}, $a^{-\frac{1}{2}}$, $a^{-\frac{2}{3}}$, can now be established as follows:

By Law I. we have

$$a^n.a^{-n} = a^0 = 1.$$

But the product of a^n and $\frac{1}{a^n}$ is also 1.

Therefore $$a^{-n} = \frac{1}{a^n}.$$

Thus $a^{-1} = \frac{1}{a}$, $a^{-3} = \frac{1}{a^3}$, $a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}}$, $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}$, $3a^{-2} = \frac{3}{a^2}$.

68. The quantity $\frac{1}{a}$ is called the *reciprocal* of a.

Thus $\frac{1}{3}$ is the reciprocal of 3, and $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$.

EXAMPLES.

1. $-2x\,(-3y)\,(+z) = - - + 6xyz = +6xyz.$

2. $-a^2\,(+a^3)\,(-a) = - + - a^{2+3+1} = +a^6.$

3. $-\frac{1}{2}x^{-1}y\,(-\frac{1}{3}x^2y^{-2})\,(-\frac{1}{4}xy^3) = - - - \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}x^{2-1+1}$ $y^{1-2+3} = -\frac{1}{24}x^2y^2.$

4. $x^{\frac{1}{2}}yz^{-\frac{1}{3}}\,(-x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{2}{3}}) = -x^{\frac{1}{2}+\frac{1}{2}}y^{1-\frac{1}{2}}z^{-\frac{1}{3}+\frac{2}{3}} = -xy^{\frac{1}{2}}z^{\frac{1}{3}}.$

EXERCISES, X.

Find the products of

1. $-7a,\ +4b,\ -c.$ 2. $+5x,\ -7x^2,\ -2x^3.$

3. $-\frac{1}{2}yz,\ +\frac{1}{3}zx,\ +xy.$ 4. $+2x^{-2}y,\ -3xy^{-3},\ 2xy.$

5. $2a^{\frac{1}{3}}b^{\frac{1}{4}},\ 5a^{\frac{2}{3}}b^{\frac{3}{4}}.$ 6. $4x^{\frac{3}{2}}y^{\frac{2}{3}},\ 3x^{-\frac{1}{2}}y^{\frac{1}{3}}.$

7. $2a^{\frac{1}{3}}b^{\frac{1}{4}},\ -a^{\frac{2}{3}}b^{\frac{1}{2}},\ ab^{\frac{1}{4}}.$

69. II. The product of two factors, one of which is a simple factor, is obtained by a law the converse of that given in Art. 31, namely,

$$\begin{aligned} +a\,(+b+c) &= + +ab + +ac = +ab+ac, \\ +a\,(-b+c) &= + -ab + +ac = -ab+ac, \\ -a\,(-b+c) &= - -ab - +ac = +ab-ac. \end{aligned}$$

Hence the product of two factors, one of which is a simple factor, is the sum of the products of the simple factor and each term of the other factor.

Thus, $-2a\,(x-y+z) = -2ax+2ay-2az;$
$-3x\,(2x^2-2x-1) = -6x^3+6x^2+3x.$

The work may be arranged as follows :

$$\begin{array}{r} 2x^2-2x-1 \\ -3x \\ \hline -6x^3+6x^2+3x. \end{array}$$

EXERCISES, XI.

Multiply

1. $2x^2-3x-1$ by $-4x.$ 2. $6x^3+x-3$ by $\frac{1}{2}x^2.$

3. $3x^2-4y^2+5z^2$ by $2x^2y.$ 4. $2x^{\frac{1}{2}}-x^{\frac{1}{4}}+2$ by $3x^{\frac{1}{4}}.$

5. x^2-xy+y^2 by $x^{-1}y^{-1}.$ 6. $\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{3}x^{-\frac{1}{4}}+1$ by $3x^{\frac{1}{2}}.$

70. III. The product of two compound factors is obtained from the law

$$(a+b)(c+d)=(a+b)c+(a+b)d=ac+bc+ad+bd,$$
$$(-a+b)(c-d)=(-a+b)c-(-a+b)d=-ac+bc+ad-bd,$$

and so on, which is the converse of the law given in Art. 54. Hence the product of two compound factors is the sum of the products of one factor and each term of the other, and may be obtained as in the following

Examples.

$$\begin{array}{llll}
x^2 & -x & -2 & \\
x & +3 & & \\
\hline
x^3 & -x^2 & -2x & \\
 & +3x^2 & -3x & -6 \\
\hline
x^3 & +2x^2 & -5x & -6.
\end{array}$$

Here x^3-x^2-2x is the product of x^2-x-2 and x, and $3x^2-3x-6$ is the product of x^2-x-2 and $+3$, and the sum of these is x^3+2x^2-5x-6, the required product.

$$\begin{array}{lllll}
x & +x^{\frac{1}{2}} & +1 & & \\
x & -x^{\frac{1}{2}} & +1 & & \\
\hline
x^2 & +x^{\frac{3}{2}} & +x & & \\
 & -x^{\frac{3}{2}} & -x & -x^{\frac{1}{2}} & \\
 & & +x & +x^{\frac{1}{2}} & +1 \\
\hline
x^2 & & +x & & +1.
\end{array}$$

$$\begin{array}{lllll}
5x^2 & -\ 2x & +\ 7 & & \\
x & +\ 3 & -\ 2x^{-1} & & \\
\hline
5x^3 & -\ 2x^2 & +\ 7x & & \\
 & +15x^2 & -\ 6x & +21 & \\
 & & -10x & +\ 4 & -14x^{-1} \\
\hline
5x^3 & +13x^2 & -9x & +25 & -14x^{-1}.
\end{array}$$

EXERCISES, XII.

Find the products of

1. $2x^2+x+1$ and $4x-3$.
2. x^2+3x-1 and x^2-3x+1.
3. x^2-xy+y^2 and $x+y$.
4. $a^4-a^2b^2+b^4$ and $a^4+a^2b^2+b^4$.
5. $x^4-x^3y+x^2y^2-xy^3+y^4$ and $x+y$.
6. $1+4x-10x^2$ and $1-6x+3x^2$.
7. x^3-7x^2+5x+1 and $2x^2-4x+1$.
8. x^3-2x^2+3x-4 and $4x^3+3x^2+2x+1$.
9. $x^2+y^2-xy+x+y-1$ and $x+y-1$.
10. $x+2y-3z$ and $x-2y+3z$.
11. $a^2+b^2+c^2-bc-ca-ab$ and $a+b+c$.
12. $a^2-2ab+b^2+c^2$ and $a^2+2ab+b^2-c^2$.
13. $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ and $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.
14. $x^{\frac{3}{4}}+y^{\frac{3}{4}}$ and $x^{\frac{3}{4}}-y^{\frac{3}{4}}$.
15. $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ and $a^{\frac{1}{3}}-b^{\frac{1}{3}}$.
16. $a^{\frac{4}{3}}-a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$ and $a^{\frac{2}{3}}+b^{\frac{2}{3}}$.
17. $x+x^{\frac{1}{2}}+2$ and $x+x^{\frac{1}{2}}-2$.
18. x^2+2+x^{-2} and x^2-2+x^{-2}.
19. x^4+x^2+1 and $x^{-4}-x^{-2}+1$.
20. $a^{-\frac{2}{3}}+a^{-\frac{1}{3}}+1$ and $a^{-\frac{1}{3}}-1$.
21. $a^{\frac{4}{3}}-2+a^{-\frac{4}{3}}$ and $a^{\frac{2}{3}}-a^{-\frac{2}{3}}$.
22. $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-x^{\frac{1}{3}}y^{\frac{2}{3}}$ and $a+a^{\frac{1}{2}}b^{\frac{1}{2}}+x^{\frac{1}{3}}y^{\frac{2}{3}}$.
23. $x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ and $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.
24. $x+\frac{1}{2}y-2$ and $\frac{4}{3}x-y+\frac{3}{2}$.
25. $\frac{1}{2}x-\frac{1}{3}+x^{-1}$ and $\frac{1}{3}x+\frac{1}{2}-x^{-1}$.
26. $x^{2p}+x^py^p+y^{2p}$ and x^p-y^p.
27. $a^m-2a^{m-1}x+3a^{m-2}x^2$ and $a^n+2a^{n-1}x-3a^{n-2}x^2$.
28. $x^{\frac{r}{2}}+2x^{\frac{r}{4}}y^p+3y^{2p}$ and $x^{\frac{r}{2}}-2x^{\frac{r}{4}}y^p+3y^{2p}$.

71. When two factors contain integral powers only of the same symbol the successive coefficients in their product may be obtained by the following method, called the method of *Detached Coefficients*, which consists in leaving out the letters in the ordinary process, and writing the coefficients only of the successive powers. Whenever a power is wanting, therefore, its place must be supplied by a zero.

Ex. 1.

$$\begin{array}{l}
x^3-3x^2+3x\ -1 \\
x^2-2x+1 \\
\hline
x^5-3x^4+\ 3x^3-\quad x^2 \\
\quad\ -2x^4+\ 6x^3-\ 6x^2+2x \\
\qquad\qquad +\quad x^3-\ 3x^2+3x-1 \\
\hline
x^5-5x^4+10x^3-10x^2+5x-1.
\end{array}$$

This product can be obtained by the above method, as follows :—

$$\begin{array}{l}
1-3+\ 3-1 \\
1-2+\ 1 \\
\hline
1-3+\ 3-\ 1 \\
\ \ -2+\ 6-\ 6+2 \\
\qquad +\ 1-\ 3+3-1 \\
\hline
1-5+10-10+5-1.
\end{array}$$

Here we have the coefficients of the successive powers in the product commencing with x^5.

Ex. 2. Find the product of $5x^3-2x+1$ and $5x^3+2x-1$.

Here the coefficient of x^2 is wanting in each factor, and its place must be supplied by 0.

$$\begin{array}{l}
\ \ 5+0-\ 2+1 \\
\ \ 5+0+\ 2-1 \\
\hline
25+0-10+5 \\
\quad +0+\ 0-0+0 \\
\qquad\ +10+0-4+2 \\
\qquad\qquad -5-0+2-1 \\
\hline
25+0+0+0-4+4-1.
\end{array}$$

Therefore the product is $25x^6-4x^2+4x-1$.

EXERCISES, XIII.

Apply the method of Detached Coefficients to find the product of

1. x^2+2x+1 and $x-1$.
2. $2x^3+3x^2+4x+5$ and x^2-2x+1
3. $6x^3-x+1$ and $6x^3+x-1$.
4. $6x^5-3x^3+1$ and $4x^4+2x^2+1$.
5. $7x^5-5x^2+2$ and $3x^3+2x^2-7x+1$.

72. The product of two like factors is called the square of either.

Thus $(a-b+c)(a-b+c)$ is the square of $a-b+c$, and may be written $(a-b+c)^2$.

When the two factors are the same, since the product of like terms in the multiplier and multiplicand is the square of either term, and the product of unlike terms in the multiplier and multiplicand is repeated in the process, it follows that *the square of any quantity is the sum of the squares of each term and twice the product of every two terms.*

For example, find the square of $a-b+c$.

$$\begin{array}{llllll}
a & -b & +c & & & \\
a & -b & +c & & & \\
\hline
a^2 & -ab & +ac & & & \\
 & -ab & & +b^2 & -bc & \\
 & & +ac & & -bc & +c^2 \\
\hline
\multicolumn{6}{l}{a^2-2ab+2ac+b^2-2bc+c^2.}
\end{array}$$

Here the products of the like terms a, a; $-b$, $-b$; $+c$, $+c$ are the squares of a, $-b$, $+c$, respectively; and the product of any two unlike terms, as $-b$, $+c$ is repeated, giving twice the product of those two, that is, $-2bc$.

EXAMPLES.

1. $(a+b-c)^2=a^2+b^2+(-c)^2+2ab+2a(-c)+2b(-c)$
$$=a+b^2+c^2+2ab-2ac-2bc.$$

In taking the products of the terms, two and two, it will be found of advantage to take in order the product of the first term, and each term that follows it, then the product of the second term, and each term that follows it, and so on, if there be more terms than three.

2. $(2x^2-3x-4)^2=4x^4+9x^2+16+2(2x^2)(-3x)$
$+2(2x^2)(-4)+2(-3x)(-4)$
$$=4x^4+9x^2+16-12x^3-16x^2+24x$$
$$=4x^4-12x^3-7x^2+24x+16.$$

3. $(x-1-x^{-1})^2=x^2+1+x^{-2}+2x(-1)+2x(-x^{-1})$
$+2(-1)(-x^{-1})$
$$=x^2+1+x^{-2}-2x-2+2x^{-1}$$
$$=x^2-2x-1+2x^{-1}+x^{-2}.$$

4. $(x^{\frac{1}{2}}-\frac{1}{2}+x^{-\frac{1}{2}})^2=x+\frac{1}{4}+x^{-1}+2x^{\frac{1}{2}}(-\frac{1}{2})+2x^{\frac{1}{2}}$
$(+x^{-\frac{1}{2}})+2(-\frac{1}{2})(+x^{-\frac{1}{2}})$
$$=x+\frac{1}{4}+x^{-1}-x^{\frac{1}{2}}+2-x^{-\frac{1}{2}}$$
$$=x-x^{\frac{1}{2}}+\frac{9}{4}-x^{-\frac{1}{2}}+x^{-1}.$$

EXERCISES, XIV.

Find the values of

1. $(1+x+x^2)^2$. 2. $(1-x+x^2)^2$ 3. $(1-x-x^2)^2$.
4. $(1-3x+2x^2)^2$. 5. $(x^2-\frac{1}{2}x+2)^2$.
6. $(x^2-2+x^{-2})^2$. 7. $(x^{\frac{3}{2}}-x^{\frac{1}{2}}+1)^2$
8. $(\frac{1}{2}x^{\frac{1}{2}}-\frac{1}{3}x^{\frac{1}{3}}-1)^2$. 9. $(a-b+c-d)^2$.
10. $(2+3x+4x^2)^2-(2-3x+4x^2)^2$.
11. $(a+b+c+d)^2-(a-b+c-d)^2$.

73. Since by actual multiplication

$$(a+b)(a-b)=a^2-b^2,$$

$$(\overline{a+b}+c)(\overline{a+b}-c)=(a+b)^2-c^2,$$

and so on, we can hence write down at once the product of two factors, one of which is the sum, and the other the difference of the same two quantities.

EXAMPLES.

1. $(2x+3y)(2x-3y)=(2x)^2-(3y)^2=4x^2-9y^2.$

2. $(2x^2+\frac{1}{2}y^{\frac{1}{2}})(2x^2-\frac{1}{2}y^{\frac{1}{2}})=(2x^2)^2-(\frac{1}{2}y^{\frac{1}{2}})^2=4x^4-\frac{1}{4}y.$

3. $(x+y+2z)(x+y-2z)=(\overline{x+y}+2z)(\overline{x+y}-2z)$

$$=(x+y)^2-(2z)^2$$

$$=x^2+y^2+2xy-4z^2.$$

Here the factors are arranged so that one is the sum, and the other the difference of $x+y$ and $2z$.

4. $(2a+b-3)(2a-b+3)=(2a+\overline{b-3})(2a-\overline{b-3})$

$$=(2a)^2-(b-3)^2$$

$$=4a^2-(b^2+9-6b)$$

$$=4a^2-b^2-9+6b.$$

Here the factors are arranged so that one is the sum and the other the difference of $2a$ and $b-3$.

5. $(x^3-x^2+x-1)(x^3-x^2-x+1)$

$$=(\overline{x^3-x^2}+\overline{x-1})(\overline{x^3-x^2}-\overline{x-1})$$

$$=(x^3-x^2)^2-(x-1)^2$$

$$=x^6+x^4-2x^5-x^2-1+2x$$

$$=x^6-2x^5+x^4-x^2+2x-1.$$

6. $(x^2-x+1-x^{-1})(x^2+x-1+x^{-1})=(x^2-\overline{x-1+x^{-1}})$ $(x^2+\overline{x-1+x^{-1}})$

$$=(x^2)^2-(x-1+x^{-1})^2$$

$$=x^4-(x^2+1+x^{-2}-2x+2-2x^{-1})$$

$$=x^4-x^2+2x-3+2x^{-1}-x^{-2}.$$

EXERCISES, XV.

Employ the preceding method to find the product of

1. $a+3b$ and $a-3b$. 2. a^2+b^2 and a^2-b^2.
3. $a^{\frac{2}{3}}+b^{\frac{2}{3}}$ and $a^{\frac{2}{3}}-b^{\frac{2}{3}}$. 4. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ and $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.
5. $4a^{\frac{1}{4}}+5b^{\frac{1}{5}}$ and $4a^{\frac{1}{4}}-5b^{\frac{1}{5}}$.
6. $\frac{1}{7}a^2+\frac{1}{9}b$ and $\frac{1}{7}a^2-\frac{1}{9}b$.
7. $\frac{a^2}{b}+\frac{c}{d^2}$ and $\frac{a^2}{b}-\frac{c}{d^2}$.
8. $2x+y+z$ and $2x+y-z$.
9. $5a-3b+c$ and $5a+3b-c$.
10. $2a^2-3b^2-4c^2$ and $2a^2+3b^2+4c^2$.
11. $x-y+z-1$ and $x-y-z+1$.
12. $2x-3y+z-1$ and $2x-3y+z+1$.
13. $2a^2-3b+c-4d$ and $2a^2+3b-c+4d$.

74. Since by actual multiplication

$$\begin{cases}(x+y)\,(x^2-xy+y^2)=x^3+y^3,\\(x-y)\,(x^2+xy+y^2)=x^3-y^3,\end{cases}$$

we can write down at once the product of any two factors one of which is the sum of two quantities, and the other the sum of the squares less the product of the same two quantities.

EXAMPLES.

1. $(2a+b)\,(4a^2-2ab+b^2)=(2a)^3+b^3=8a^3+b^3$.

Here the second factor is the sum of the squares of $2a$ and b less their product.

2. $(x-3y)\,(x^2+3xy+9y^2)=x^3-(3y)^3=x^3-27y^3$.

Here the second factor is the sum of the squares of x and $-3y$ less their product.

3. $(x+x^{-1})\ \ (x^2-1+x^{-2})=(x+x^{-1})\ \ (x^2+x^{-2}-1)$ $=x^3+x^{-3}$.

4. $(x^3-y^2)(x^6+x^3y^2+y^4)=(x^3)^3-(y\ )^3=x^9-y^6$.

EXERCISES, XVI.

Find the product of

1. $2x+3y$ and $4x^2-6xy+9y^2$.
2. $x-2y$ and $x^2+2xy+4y^2$.
3. a^2-b and $a^4+a^2b+b^2$
4. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ and $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.
5. $2x^{\frac{1}{2}}-3y^{\frac{1}{2}}$ and $4x+6x^{\frac{1}{2}}y^{\frac{1}{2}}+9y$.
6. x^p+y^q and $x^{2p}-x^py^q+y^{2q}$.
7. $ax^{-1}+a^{-1}x$ and $a^2x^{-2}-1+a^{-2}x^2$.

DIVISION.

75. Any factor of a product is said to be the *quotient* of the product divided by the other factor.

Thus $+2y$ is the quotient of $-3x\,(+2y)$, or $-6xy$ divided by $-3x$; $2x-1$ is the quotient of $(2x-1)(x+2)$, or $2x^2+3x-2$, divided by $x+2$.

76. The quotient of one quantity divided by another is denoted by enclosing them in brackets and writing the sign ÷ between them, or by writing the second below the first with a line between them.

Thus the quotient of $-a$ divided by $+b$ is denoted by $(-a)\div(+b)$, or by $\frac{-a}{+b}$; the quotient of $2x^2-3x+1$ divided by $x-1$ is denoted by $(2x^2-3x+1)\div(x-1)$, or by $\frac{2x^2-3x+1}{x-1}$; and so on.

77. The first or upper quantity is called the *Dividend*, and the second or lower one the *Divisor*.

78. When the Dividend and Divisor are mononomials the brackets are generally omitted.

Thus $(-2x)\div(+3y)$ is written $-2x\div+3y$.

79. Since Division is the inverse of Multiplication, it will be found convenient to adopt the course pursued in the last Chapter, and consider in order three cases, namely—

I. The Division of mononomials.

II. The Division of a polynomial by a mononomial.

III. The Division of polynomials.

80. I. The Rule of Signs in Division may be deduced as follows:—

Since $\left(+\frac{a}{b}\right)(+b)=+a$, it follows from Art. 75 that $\frac{+a}{+b}=+\frac{a}{b}$. In the same manner, since $\left(+\frac{a}{b}\right)(-b)=-a$, $\left(-\frac{a}{b}\right)(+b)=-a$, $\left(-\frac{a}{b}\right)(-b)=+a$, it follows that $\frac{-a}{-b}=+\frac{a}{b}$, $\frac{-a}{+b}=-\frac{a}{b}$, and $\frac{+a}{-b}=-\frac{a}{b}$.

Hence the Rule of Signs is the same in Division as in Multiplication.

Thus $\frac{+4x}{-3y}=-\frac{4x}{3y}$, $\frac{-x^2}{+3y^2}=-\frac{x^2}{3y^2}$, $\frac{-2x}{-3}=+\frac{2x}{3}$.

81. When the Dividend and Divisor involve powers of the same quantity, the quotient can be obtained according to the Law

$$\frac{a^m}{a^n}=a^{m-n},$$

m and n being any real quantities whatsoever.

This Law follows from the first Index Law, thus—

$$a^{m-n}\,a^n=a^m;$$

$$\therefore \frac{a^m}{a^n}=a^{m-n}.$$

Thus $\frac{a^3}{a^2}=a^{3-2}=a,\quad \frac{a^2}{a^4}=a^{2-4}=a^{-2};\quad \frac{a^2}{a^{-3}}=a^{2+3}=a^5,$

$$\frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}}=a^{\frac{1}{2}-\frac{1}{3}}=a^{\frac{1}{6}},\quad \frac{a^2}{a^2}=a^{2-2}=a^0=1.$$

EXAMPLES.

1. $\frac{-2x^2}{+x}=-\frac{2x^2}{x}=-2x.$

2 $\frac{-4x^2y^3}{-xy^2}=+\frac{4x^2y^3}{xy^2}=+4xy.$

3. $\frac{x^{\frac{1}{2}}z^{\frac{2}{3}}}{+x^{-\frac{1}{2}}z^{\frac{1}{3}}}=+xz^{\frac{1}{3}}.$

4. $\frac{-a^ab^b}{+a^bb^a}=-a^{a-b}b^{b-a}.$

EXERCISES, XVII.

Divide

1. $-2x^2$ by $+x$.
2. $+6x^2y^5$ by $-2xy^3$.
3. $-x^{\frac{3}{2}}y$ by $x^{\frac{1}{2}}y$.
4. $+xy^2z^3$ by $-x^{-1}yz^2$.
5. $2a^{\frac{1}{4}}$ by $-a^{-\frac{3}{4}}$.
6. $-5x^{-2}y^3$ by $+x^{-3}y^2$.
7. $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$ by $a^{\frac{1}{3}}b^{\frac{1}{4}}c^{\frac{1}{5}}$.

82. II. Since $\left(+\frac{a}{b}-\frac{c}{b}\right)(+b)=+a-c$, it follows that

$$\frac{+a-c}{+b}=+\frac{a}{b}-\frac{c}{b}.$$

In the same manner it may be shown that

$$\frac{+a-b+c-d}{-e}=-\frac{a}{e}+\frac{b}{e}-\frac{c}{e}+\frac{d}{e},$$

and generally that the quotient of a polynomial by a mononomial is obtained by dividing each term of the former by the latter.

EXAMPLES.

1. $\dfrac{4x^2-2x}{-x}=-4x+2.$

2. $\dfrac{ax^3+a^2x^2-a^3x}{ax}=x^2+ax-a^2.$

EXERCISES, XVIII.

Divide

1. $48x^4-16x^3+24x^2$ by $8x^2$.
2. $15ax^3-25bx^2y+35cxy^2$ by $-5x$.
3. $m^2nx^4y^2-mn^2x^2y^4$ by $mnxy$.
4. $a^2bx^{2m+n}-ab^2x^{m+2n}$ by abx^{m+n}.
5. $5x^2y^{-1}-6x^{-1}y^2$ by $-x^{-1}y^{-1}$.
6. $mx^{m+2}y^2-nx^2y^{n+2}-x^2y^2$ by x^2y^2.

83. III. Since $(a+b)(c+d)=(a+b)c+(a+b)d$ $=ac+bc+ad+bd$, it follows that

$$\frac{ac+bc+ad+bd}{a+b}=\frac{(a+b)c+(a+b)d}{a+b}=c+d.$$

In the same manner it may be shown that

$$\frac{x^2+3x+2}{x+1}=\frac{x^2+x+2x+2}{x+1}=\frac{x(x+1)+2(x+1)}{x+1}$$
$$=x+2.$$

$$\frac{2x^2+5x-12}{x+4}=\frac{2x^2+8x-3x-12}{x+4}=\frac{2x(x+4)-3(x+4)}{x+4}$$
$$=2x-3.$$

$$\frac{6x^3-5x^2-3x+2}{3x+2}=\frac{6x^3+4x^2-9x^2-6x+3x+2}{3x+2}$$
$$=\frac{2x^2(3x+2)-3x(3x+2)+(3x+2)}{3x+2}$$
$$=2x^2-3x+1.$$

And so generally the successive terms in the quotient may be determined by arranging the dividend as the sum of certain quantities, each of which contains the divisor as a factor.

Thus in the last example the dividend is arranged as the sum of $2x^2(3x+2)$, $-3x(3x+2)$, and $+(3x+2)$, each of which is exactly divisible by the divisor $3x+2$.

84. The preceding method may be equivalently replaced by writing the dividend and divisor according to the descending or ascending powers of x and proceeding as follows:

$$\begin{array}{l} 3x+2)\ 6x^3-5x^2-3x+2\ (2x^2 \\ \qquad\qquad \underline{6x^3+4x^2\qquad\;} \\ \qquad\qquad\qquad -9x^2-3x+2. \end{array}$$

Here the first term in the quotient $2x^2$ is obtained by dividing $6x^3$, the first term in the dividend, by $3x$, the first term in the divisor. The first term in the quotient is then multiplied into $3x+2$, giving $6x^3+4x^2$, and this subtracted from the dividend leaves $-9x^2-3x+2$ which is the product of the divisor and the remaining terms of the quotient.

The next term $-3x$ will be obtained in like manner from $3x+2$ and $-9x^2-3x+2$.

$$\begin{array}{l} 3x+2)-9x^2-3x+2\,(-3x \\ \qquad\qquad \underline{-9x^2-6x\qquad} \\ \qquad\qquad\qquad\qquad 3x+2. \end{array}$$

The product of the divisor and the second term of the quotient is here subtracted from $-9x^2-3x+2$, leaving $3x+2$, which must be the product of the divisor, and the third term of the quotient.

The third term of the quotient $+1$ is found in like manner from the divisor $3x+2$ and the last difference $3x+2$, thus,

$$\begin{array}{l} 3x+2)\ 3x+2\ (+1 \\ \qquad\qquad \underline{3x+2} \end{array}$$

The different steps in the preceding process may be collected and arranged as follows:

$$\begin{array}{rl}
3x+2) & 6x^3-5x^2-3x+2\ (2x^2-3x+1 \\
 & \underline{6x^3+4x^2} \\
 & \quad -9x^2-3x \\
 & \quad \underline{-9x^2-6x} \\
 & \qquad\qquad 3x+2 \\
 & \qquad\qquad \underline{3x+2}
\end{array}$$

Here the divisor is written only once, and two terms only of the first difference are brought down.

85. The same result may be obtained by arranging the dividend and divisor according to ascending powers of x, as follows:

$$\begin{array}{rl}
2+3x) & 2-3x-5x^2+6x^3\ (1-3x+2x^2 \\
 & \underline{2+3x} \\
 & \quad -6x-5x^2 \\
 & \quad \underline{-6x-9x^2} \\
 & \qquad\qquad 4x^2+6x^3 \\
 & \qquad\qquad \underline{4x^2+6x^3}
\end{array}$$

EXAMPLES.

1. Divide x^4-a^4 by $x-a$.

$$\begin{array}{rl}
x-a) & x^4-a^4\ (x^3+ax^2+a^2x+a^3 \\
 & \underline{x^4-ax^3} \\
 & \quad ax^3 \\
 & \quad \underline{ax^3-a^2x^2} \\
 & \qquad a^2x^2 \\
 & \qquad \underline{a^2x^2-a^3x} \\
 & \qquad\qquad a^3x-a^4 \\
 & \qquad\qquad \underline{a^3x-a^4}
\end{array}$$

Here $-a^4$ is not brought down until the last step.

2. Divide x^2+1+x^{-2} by $x-1+x^{-1}$

In this case $+1$ may be considered the coefficient of x^0, and therefore the terms are arranged according to the descending powers of x, the successive indices being $2,\ 0,\ -2$.

$$\begin{array}{l}
x-1+x^{-1})\,x^2+1+x^{-2}\,(x+1+x^{-1}\\
\qquad\qquad\quad \underline{x^2-x+1}\\
\qquad\qquad\qquad x+x^{-2}\\
\qquad\qquad\qquad \underline{x-1+x^{-1}}\\
\qquad\qquad\qquad\qquad 1-x^{-1}+x^{-2}\\
\qquad\qquad\qquad\qquad \underline{1-x^{-1}+x^{-2}}
\end{array}$$

Here the third term of the quotient x^{-1} is the quotient of 1 or x^0 by x.

3. Divide $2x^3+\frac{3}{2}x^{\frac{5}{2}}-\frac{1}{2}x^2-\frac{63}{40}x^{\frac{3}{2}}+\frac{3}{5}x+\frac{1}{2}x^{\frac{1}{2}}-\frac{4}{5}$ by $2x^{\frac{3}{2}}-\frac{1}{2}x^{\frac{1}{2}}+\frac{4}{5}$.

$$\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Big(x^{\frac{3}{2}}+\frac{3}{4}x-1\\
2x^{\frac{3}{2}}-\frac{1}{2}x^{\frac{1}{2}}+\frac{4}{5}\Big)\,2x^3+\frac{3}{2}x^{\frac{5}{2}}-\frac{1}{2}x^2-\frac{63}{40}x^{\frac{3}{2}}+\frac{3}{5}x+\frac{1}{2}x^{\frac{1}{2}}-\frac{4}{5}\\
\qquad\qquad\qquad \underline{2x^3-\frac{1}{2}x^2+\frac{4}{5}x^{\frac{3}{2}}}\\
\qquad\qquad\qquad\quad \frac{3}{2}x^{\frac{5}{2}}-\frac{95}{40}x^{\frac{3}{2}}+\frac{3}{5}x\\
\qquad\qquad\qquad\quad \underline{\frac{3}{2}x^{\frac{5}{2}}-\frac{3}{8}x^{\frac{3}{2}}+\frac{3}{5}x}\\
\qquad\qquad\qquad\qquad\quad -2x^{\frac{3}{2}}\qquad+\frac{1}{2}x^{\frac{1}{2}}-\frac{4}{5}\\
\qquad\qquad\qquad\qquad\quad \underline{-2x^{\frac{3}{2}}\qquad+\frac{1}{2}x^{\frac{1}{2}}-\frac{4}{5}}
\end{array}$$

4. Divide $2a^5-6a^3b+13a^2b^2-6ab^3-3a^4b$ by $2a-3b$.

Here we shall arrange the dividend and divisor according to descending powers of a.

$$\begin{array}{rl}
2a-3b) & 2a^5-3ba^4-6ba^3+13b^2a^2-6b^3a\,(a^4-3ba^2+2b^2a \\
 & \underline{2a^5-3ba^4} \\
 & \qquad\quad -6ba^3+13b^2a^2 \\
 & \qquad\quad \underline{-6ba^3+9b^2a^2} \\
 & \qquad\qquad\qquad 4b^2a^2-6b^3a \\
 & \qquad\qquad\qquad \underline{4b^2a^2-6b^3a}
\end{array}$$

5. Divide $a^4-ab^3-ac^3-2ba^3+2b^4+2bc^3+3ca^3-3cb^3-3c^4$ by $a+3c-2b$.

Here the dividend will be arranged according to powers of a, the coefficients of a^3 and of a being collected within brackets.

$$\begin{array}{rl}
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a^3-(c^3+b^3) \\
a+3c-2b) & a^4+(3c-2b)a^3-(c^3+b^3)a-3c^4+2bc^3-3b^3c+2b^4 \\
 & \underline{a^4+(3c-2b)a^3} \\
 & \qquad\qquad -(c^3+b^3)a-3c^4+2bc^3-3b^3c+2b^4 \\
 & \qquad\qquad \underline{-(c^3+b^3)a-3c^4+2bc^3-3b^3c+2b^4}
\end{array}$$

Exercises, XIX.

Divide

1. $2-3x+x^3$ by $2+x$.
2. $10x^3+7x^2-8x+6$ by $2x+3$.
3. x^3+1 by $x+1$.
4. $2x^3-x^2+3x-9$ by $2x-3$.
5. $6x^3+14x^2-4x+24$ by $2x+6$.
6. $7x^3-24x^2+58x-21$ by $7x-3$.
7. x^5-1 by $x-1$.
8. $x^4-5x^3+11x^2-12x+6$ by x^2-3x+3.
9. x^4-13x^2+36 by x^2+5x+6.
10. $1-x-3x^2-x^5$ by $1+2x+x^2$.
11. x^6+x^4-2 by x^4+2x^2+2.
12. $x^8+2x^6+3x^4+2x^2+1$ by $x^4-2x^3+3x^2-2x+1$.
13. $\frac{3}{4}x^5-4x^4+\frac{77}{8}x^3-\frac{43}{4}x^2-\frac{33}{4}x+27$ by $\frac{1}{2}x^2-x+3$.
14. x^3+x^{-3} by by $x+x^{-1}$.
15. $2x^2-3x-3+7x^{-1}-3x^{-2}$ by $2x+1-3x^{-1}$.
16. y^5-y^{-5} by $y-y^{-1}$.

17. x^2-2+x^{-2} by $x-2+x^{-1}$.

18. x^2+64 by $x+4x^{\frac{1}{2}}+8$.

19. $x^2+x^{\frac{3}{2}}-9x-16x^{\frac{1}{2}}-4$ by $x+4x^{\frac{1}{2}}+4$.

20. $x^5+2x^4y+3x^3y^2-x^2y^3-2xy^4-3y^5$ by x^3-y^3.

21. a^6-b^6 by $a^3-2a^2b+2ab^2-b^3$.

22. $a^4-3b^4-2b^2a^2+a^2+b^2$ by a^2+b^2.

23. $x^4+2nx^2-m^2x^2+n^2$ by x^2+mx+n.

24. $a^2-b^2-c^2-2bc$ by $a-b-c$.

25. $x^4+m^3x-2mnx+m^2n-n^2$ by x^2+mx+n.

26. $a^4+2a^2b^2+9b^4$ by $a^2+2ab+3b^2$.

27. $x^3-3xy-y^3-1$ by $x-y-1$.

28. $x^3-(a+b+c)x^2+(bc+ca+ab)x-abc$ by $x^2-(a+b)x+ab$.

29. $x^3+y^3+z^3-3xyz$ by $x+y+z$.

30. x^2-y by $x^{\frac{1}{2}}-y^{\frac{1}{4}}$.

31. a^3-b^2 by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.

32. $x^{\frac{3}{2}}-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

33. $x-y$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

34. $a^2+2ab^{-1}+9b^{-2}$ by $a-2a^{\frac{1}{2}}b^{-\frac{1}{2}}+3b^{-1}$.

35. $x^2-2x^{\frac{3}{2}}y^{-\frac{1}{2}}+2xy^{-1}-x^{\frac{1}{2}}y^{-\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{-\frac{1}{2}}$.

36. $a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{-\frac{1}{3}}+b^{-\frac{2}{3}}$ by $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{-\frac{1}{6}}+b^{-\frac{1}{3}}$.

37. $x^{\frac{3}{2}}-2a^{\frac{3}{2}}x^{\frac{3}{4}}+a^3$ by $x^{\frac{1}{2}}-2a^{\frac{1}{2}}x^{\frac{1}{4}}+a$.

38. $x-4x^{\frac{3}{4}}y^{-\frac{1}{8}}+6x^{\frac{1}{2}}y^{-\frac{1}{4}}-4x^{\frac{1}{4}}y^{-\frac{3}{8}}+y^{-\frac{1}{2}}$ by $x^{\frac{1}{2}}-2x^{\frac{1}{4}}y^{-\frac{1}{8}}+y^{-\frac{1}{4}}$.

86. When there is no exact quotient the process of the preceding Article will give at any stage a quantity which, multiplied into the divisor, and added to the last difference, will be equal to the dividend.

For example,

$$\begin{array}{rl} x-2\,)\,2x^2-6x+7\,(\,2x & \\ \underline{2x^2-4x} & \\ -2x+7. & \end{array}$$

Here $2x(x-2)-2x+7=2x^2-6x+7$.

If the process be continued, the next term of the quotient being -2, and the next difference 3, it follows that

$$(2x-2)(x-2)+3=2x^2-6x+7.$$

87. The statement of the preceding Article is true, no matter what the arrangement of the terms of the dividend and divisor is.

Thus, in dividing $x^4-3x^3+4x^2-7x+6$ by x^2-2x+3, we may proceed as follows:

$$\begin{array}{r} -2x+x^2-3\,)\,-3x^3+x^4-7x+4x^2+6\,(\,\tfrac{3}{2}x^2+\tfrac{1}{4}x^3 \\ \underline{-3x^3+\tfrac{3}{2}x^4\qquad-\tfrac{9}{2}x^2\qquad\qquad} \\ -\tfrac{1}{2}x^4-7x+\tfrac{17}{2}x^2+6\qquad\qquad \\ \underline{-\tfrac{1}{2}x^4+\tfrac{1}{4}x^5-\tfrac{3}{4}x^3\qquad\qquad\qquad} \\ -\tfrac{1}{4}x^5+\tfrac{3}{4}x^3+\tfrac{17}{2}x^2-7x+6\qquad \end{array}$$

If we stop at this stage of the process it will follow that

$$\left(\tfrac{3}{2}x^2+\tfrac{1}{4}x^3\right)\left(-2x+x^2-3\right)-\tfrac{1}{4}x^5+\tfrac{3}{4}x^3+\tfrac{17}{2}x^2-7x+6$$
$$=x^4-3x^3+4x^2-7x+6.$$

88. When the dividend and divisor are arranged according to descending positive powers of the same letter, it is always possible to continue the process of division until the leading term of the last difference is of a lower degree than the leading term of the divisor.

The last difference in such cases is called the Remainder.

EXAMPLES.

1. Find the quotient and remainder when $10x^3+7x^2-8x+14$ is divided by $2x+3$.

$$\begin{array}{l}
2x+3\,)\,10x^3+7x^2-8x+14\,(\,5x^2-4x+2 \\
\qquad\quad \underline{10x^3+15x^2} \\
\qquad\qquad\quad -8x^2-8x \\
\qquad\qquad\quad \underline{-8x^2-12x} \\
\qquad\qquad\qquad\qquad 4x+14 \\
\qquad\qquad\qquad\qquad \underline{4x+6} \\
\qquad\qquad\qquad\qquad\quad 8
\end{array}$$

Hence the quotient $=5x^2-4x+2$, and remainder $=8$.

2. Find the quotient and remainder when x^2+px+q is divided by $x-a$.

$$\begin{array}{l}
x-a\,)\,x^2+px+q\,(\,x+a+p \\
\qquad\quad \underline{x^2-ax} \\
\qquad\quad (a+p)x+q \\
\qquad\quad \underline{(a+p)x-a(a+p)} \\
\qquad\qquad\qquad a(a+p)+q.
\end{array}$$

Hence the quotient $=x+a+p$, and remainder $=a(a+p)+q$.

EXERCISES, XX.

Find the quotient and remainder in dividing

1. $4x^3-4x^2+8x+2$ by $2x+1$.
2. x^2+a^2 by $x+a$.
3. x^3-a^3 by $x+a$.
4. $2x^5-2x^4+9x^3$ by $2x^3+x+1$.
5. $2x^5+2x^4+5x^3$ by x^3+x^2+x+1.
6. $2x^4+3x^3y+3x^2y^2+3xy^3+y^4+2y^5$ by $2x+y$.
7. x^3+3ax^2 by $x^2+5ax+a^2$.

HORNER'S METHOD OF DIVISION.

89. When the dividend and divisor contain integral powers only of the same symbol, and the coefficient of the first term of the divisor is unity, the quotient may be found by the following process, which is the inverse of Multiplication by Detached Coefficients.

By multiplication the product of $3x^2-x+2$ and x^2-2x+3 is obtained as follows:

$$\begin{array}{rrrrr}
3 & -1 & +\ 2 & & \\
1 & -2 & +\ 3 & & \\
\hline
3 & -1 & +\ 2 & & \\
 & -6 & +\ 2 & -4 & \\
 & & +\ 9 & -3 & +6 \\
\hline
3 & -7 & +13 & -7 & +6.
\end{array}$$

Therefore the product is $3x^4-7x^3+13x^2-7x+6$.

Here, since the sum of the third, fourth, and fifth horizontal rows of coefficients is equal to the sixth, it follows that the third row is equal to the sum of the fourth and fifth with their signs changed, and the sixth.

Thus
$$\begin{array}{rrrrr}
3 & -7 & +13 & -7 & +6 \\
 & +6 & -\ 2 & +4 & \\
 & & -\ 9 & +3 & -6 \\
\hline
3 & -1 & +\ 2. & &
\end{array}$$

Consequently the quotient of $3x^4-7x^3+13x^2-7x+6$ divided by x^2-2x+3 may be obtained as follows.

$$\begin{array}{r|rrrrr}
 & 3 & -7 & +13 & -7 & +6 \\
+2 & & +6 & -\ 2 & +4 & \\
-3 & & & -\ 9 & +3 & -6 \\
\hline
 & 3 & -1 & +5 & &
\end{array}$$

2

In forming the product above it will be noticed that the diagonal columns

$$\begin{array}{cccccc} -6 & & +2 & & -4 & \\ & +9 & & -3 & & +6 \end{array}$$

are the products of $-2+3$ and $3, -1, +2$, respectively. Therefore in performing the inverse process of finding the quotient the diagonal columns

$$\begin{array}{cccccc} +6 & & -2 & & +4 & \\ & -9 & & +3 & & -6 \end{array}$$

are the products of $+2\ -3$ and $3, -1, +2$, respectively. $+2, -3$ are the coefficients of the second and third terms of the divisor with their signs changed, and are written vertically, each horizontally opposite the products of which it is a factor.

Ex. 2. Multiply $2x^2-3x+2$ by x^2-2x+4, and divide the product by x^2-2x+4.

$$\begin{array}{r|rrrrr}
\multicolumn{1}{r}{} & 2 & -3 & +2 & & \\
\multicolumn{1}{r}{} & 1 & -2 & +4 & & \\ \cline{2-4}
\multicolumn{1}{r}{} & 2 & -3 & +2 & & \\
\multicolumn{1}{r}{} & & -4 & +6 & -4 & \\
\multicolumn{1}{r}{} & & & +8 & -12 & +8 \\ \cline{2-6}
 & 2 & -7 & +16 & -16 & +8 \\
+2 & & +4 & -6 & +4 & \\
-4 & & & -8 & +12 & -8 \\ \cline{2-6}
\multicolumn{1}{r}{} & 2 & -3 & +2 & &
\end{array}$$

Here in the inverse process $+2, -4$ the coefficients of the second and third terms of the divisor, with their signs changed, are written in a vertical line, and the products of these and the successive coefficients of the quotient are written in diagonal columns. The first coefficient of the quotient is the same as the first coefficient of the dividend, the coefficient of the divisor being unity.

Ex. 3. Multiply $3x^3-2x+5$ by x^3-2x^2-1 and divide the product by x^3-2x^2-1.

$$\begin{array}{rrrrrrr}
3 & +0 & -2 & +5 & & & \\
1 & -2 & +0 & -1 & & & \\
\hline
3 & +0 & -2 & +5 & & & \\
 & -6 & -0 & +4 & -10 & & \\
 & & +0 & +0 & -\ 0 & +0 & \\
 & & & -3 & -\ 0 & +2 & -5 \\
\hline
\end{array}$$

$$\begin{array}{r|rrrrrrr}
 & 3 & -6 & -2 & +6 & -10 & +2 & -5 \\
+2 & & +6 & +0 & -4 & +10 & & \\
-0 & & & -0 & -0 & +\ 0 & -0 & \\
+1 & & & & +3 & +\ 0 & -2 & +5 \\
\hline
 & 3 & +0 & -2 & +5 & +\ 0 & +0 & +0
\end{array}$$

Here zeros are supplied where any terms are wanting.

The process of division exhibited in the above examples, as the inverse of multiplication, is called Horner's Method of Division. The following additional examples will exhibit the mode of conducting the various steps in the process.

Ex. 4. Divide $2x^5+7x^4+20x^3+30x^2+34x+35$ by x^2+2x+5.

$$\begin{array}{r|rrrrrr}
 & 2 & +7 & +20 & +30 & +34 & +35 \\
-2 & & -4 & -\ 6 & -\ 8 & -14 & \\
-5 & & & -10 & -15 & -20 & -35 \\
\hline
 & 2 & +3 & +\ 4 & +\ 7 & +\ 0 & +\ 0
\end{array}$$

Therefore the quotient is $2x^3+3x^2+4x+7$.

Here the coefficients of the dividend are written in a horizontal line, and the coefficients of the divisor, except the first, with their signs changed, in a vertical line. The first coefficient 2 in the dividend is repeated below as the first coefficient in the quotient, and the products of this 2 and -2, -5, give the first diagonal column to the right of the divisor, namely,

$$\begin{array}{rr}
-4 & \\
 & -10
\end{array}$$

The sum of the second vertical column to the right of the divisor gives $+3$, the second coefficient in the quotient, and the products of this $+3$ and -2, -5, give the second diagonal column,

$$\begin{array}{ll} -6 & \\ & -15 \end{array}$$

The sum of the third vertical column gives $+4$, the third coefficient in the quotient, and the products of this $+4$ and -2, -5, constitute the third diagonal column,

$$\begin{array}{ll} -8 & \\ & -20 \end{array}$$

The process is continued in this manner and ceases when the last product, (in the above case -35,) falls vertically below the last term of the dividend.

The successive coefficients of the quotient are thus found, and as the first terms of the dividend and divisor are of the fifth and second order, respectively, the first term of the quotient must be of the third order.

Therefore the quotient is

$$2x^3+3x^2+4x+7.$$

If the dividend in the last example had been

$$2x^5+7x^4+20x^3+30x^2+33x+39,$$

that is,

$$2x^5+7x^4+20x^3+30x^2+34x+35-x+4,$$

and the divisor the same as before, the remainder would evidently be $-x+4$. The sucessive coefficients -1, $+4$, of this remainder will therefore be the sums of the last two vertical columns in the following, the process being conducted as before.

$$\begin{array}{r|rrrrrr} & 2 & +7 & +20 & +30 & +33 & +39 \\ -2 & & -4 & -6 & -8 & -14 & \\ -5 & & & -10 & -15 & -20 & -35 \\ \hline & 2 & +3 & +4 & +7 & -1 & +4 \end{array}$$

In such a case as this the last multiplier for forming the diagonal columns will be the last coefficient of the quotient, and the sums of the vertical columns to the right of this multiplier will be the successive coefficients of the remainder. The order of any term in the remainder is the same as that term of the dividend in the same vertical column. Thus in the last example -1 is the coefficient of x, as is also $+33$ in the dividend.

It is usual to draw a vertical line as above between the coefficients of the quotient and those of the remainder.

Ex. 5. Divide $10x^4-7x^3-19x^2$ by $x-3$.

Here the places of the coefficients of the two terms wanting in the dividend must be supplied by zeros.

$$\begin{array}{r|rrrr|r}
 & 10 & -\ 7 & -19 & +\ \ 0 & +\ \ 0 \\
+3 & & +30 & +69 & +150 & +450 \\ \hline
 & 10 & +23 & +50 & +150 & +450
\end{array}$$

Therefore, quotient $=10x^3+23x^2+50x+150$, and remainder $=450$.

Ex. 6. Divide $5x^5-18x^3-8x^2+20x-5$ by x^3+2x^2-3.

In this case the place of the term wanting in the divisor is supplied by a zero.

$$\begin{array}{r|rrr|rrr}
 & 5 & +\ 0 & -18 & -\ 8 & +20 & -5 \\
-2 & & -10 & +20 & -\ 4 & & \\
+0 & & & +\ 0 & -\ 0 & +0 & \\
+3 & & & & +15 & -30 & +6 \\ \hline
 & 5 & -10 & +\ 2 & +\ 3 & -10 & +1
\end{array}$$

Here the operation of forming the coefficients of the quotient ceases as soon as the last product $+6$ falls below the last term of the dividend. A vertical line is then drawn after $+2$ the last multiplier, and the sums of the vertical columns to the right of this line will be the successive coefficients of the remainder.

Therefore, quotient $= 5x^2 - 10x + 2$, remainder $= 3x^2 - 10x + 1$.

EXERCISES, XXI.

Divide according to Horner's Method

1. $5x^3 - 4x^2 - 3x - 90$ by $x - 3$.
2. $x^3 + 3x^2 + 3x + 1$ by $x + 1$.
3. $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$ by $x^2 - 2x + 1$.
4. $20x^6 - 20x^5 + 57x^4 + 3x^3 - 7x^2 - 2x + 6$ by $x^2 - x + 3$.
5. $17x^5 - 15x^4 - 304$ by $x - 2$.
6. $x^9 + x^8 + x^7 + 2x^6 - x^4 - x^2 - 2x - 1$ by $x^4 + x^3 + x + 1$.
7. $4x^5 - 7x^4 + 25x^3 - 15x^2 + 8x + 10$ by $x^2 - x + 5$.
8. $11x^4 - 2x^3 + 14x - 539$ by $x^2 - 7$.
9. $5x^4 - 7x^3 + 15x^2 - 12x - 36$ by $x^2 + x + 3$.
10. $4x^4 - 18x^3 - 12x^2 + 13x$ by $x - 5$.
11. $7x^5 + 19x^4 - 5x^3$ by $x + 3$.
12. $7x^4 - 3x^2 + 1$ by $x^2 - 2x + 3$.
13. $10x^5 + 5x^4 - 90x^3 - 44x^2 + 10x + 1$ by $x^2 - 9$.
14. $5x^5 - 15x^3 + 20x + 42$ by $x^3 - 2x^2 + 3$.
15. $8x^4 - 400x^2 + 176x$ by $x^3 + 7x^2 - 3x + 1$.
16. $10x^{10} + 10x^6 + 10x^3 - 100$ by $x^7 + x^3 - x + 1$.
17. $3x^5 + 5x^4 - 3x^3 + 7x^2 - + 8$ by $x^2 - 2x$.
18. $9x^{13} + 4x^6 - 27x^5 + 1$ by $x^8 + 2x^4 + 1$.

MISCELLANEOUS THEOREMS.

90. Any quantity is said to be a *function* of the symbol or symbols involved in it.

Thus $5x$, $x^2 - 1$, $5x^3 - x + 1$, $x^3 - y^2 - x + y$ are functions of x, the latter being also a function of y.

91. A function of any quantity x is denoted by enclosing x in a bracket, and writing before it any one of the symbols f, ϕ, ψ, &c., either alone or affected with accents, dashes, or subscripts.

Thus $f(x)$, $f_1(x)$, $f'(x)$, $f_2(x)$, $\phi(x)$, $\psi(x)$ denote different quantities involving x.

92. It must be carefully noted in this notation that such symbols as f, ϕ, employed before the brackets do not denote numbers, but are merely short forms for expressing the words *quantity*, *expression*, or *function involving*. Such a symbol, therefore, as $f(y)$ must not be confounded with the product of f and y.

93. If a certain quantity involving x be denoted by $f(x)$, then the value of the same quantity when $x=a$ is denoted by $f(a)$.

Thus, if $f(x) = x^3-5x^2+10$, it will follow that

$$\begin{aligned}
f(a) &= a^3-5a^2+10,\\
f(b) &= b^3-5b^3+10,\\
f(-y) &= -y^3-5y^2+10,\\
f(c^2) &= c^6-5c^4+10,\\
f(2x) &= 8x^3-20x^2+10,\\
f(1) &= 1-5+10=6,\\
f(-3) &= -27-45+10=-62,\\
f(0) &= 10,
\end{aligned}$$

and so on.

Examples.

1. If $f(x)=x^2-2x+3$ and $\phi(x)=2x^3-3x^2-5$, find the value of $f(1)+\phi(2)$.

Here $f(1)=1-2+3=2$,

$\phi(2)=16-12-5=-1$.

Therefore $f(1)+\phi(2)=2-1=1$.

2. If $f(x)=5x^2-2xy+y^2$, find $f(y)$.

Here $f(y)$ will be the value $5x^2-2xy+y^2$ when for x we write y.

Therefore $f(y)=5y^2-2y^2+y^2=4y^2$.

3. If $f_1(x)=x^2-2xy-y^2$, and $f_2(y)=y^2+2yx-x^2$, find the value of $f_2(x)-f_1(y)$.

$$f_2(x)=x^2+2x^2-x^2=2x^2,$$
$$f_1(y)=y^2-2y^2-y^2=-2y^2.$$

Therefore $f_2(x)-f_1(y)=2x^2+2y^2$.

Exercises, XXII.

1. If $f(x)=x^3-2x^2+6x-10$, find $f(2)$.

2. If $f(a)=5a^2+6a-20$, find $f(-3)$.

3. If $f(y)=9y^3-3y^2-1$, find $f(\frac{2}{3})$.

4. If $f(x)=5x^3-3x+10$, and $\phi(y)=3y^2-5y-15$, find $f(-1)-\phi(-2)$.

5. If $f_1(a)=a^2-b^2-c^2$, and $f_2(b)=b^2-c^2-a^2$, find $f_2(1)-f_1(1)$.

6. If $f(x)=2x+y-z$, $\phi(y)=x-2y-z$, and $\psi(z)=x+y+2z$, find $f(1)+\phi(2)-\psi(-1)$.

94. If it be required to express in the above notation that a quantity involves two letters, as x and y, we employ the symbol $f(x, y)$. In such a case the value of the quantity when $x=a$, and $y=b$ is written $f(a, b)$.

Thus, if $f(x, y)=2x^2-xy+3y^2$, then

$$f(a, b)=2a^2-ab+3b^2,$$
$$f(x, 1)=2x^2-x+3,$$
$$f(a, 2)=2a^2-2a+12.$$

95. When the divisor contains the first power only of the symbol according to whose powers the dividend

and divisor are arranged, the remainder may be found without actual division in the following manner:

Let $f(x)$ denote the dividend, $ax+b$ the divisor, Q the quotient, and R the remainder, which in this case will not contain x, because the divisor involves only the first power of x.

Then, since the product of the quotient and divisor added to the remainder is equal to the dividend, we have

$$f(x)=Q(ax+b)+R.$$

Now since this equality holds always, it will be true when x has the value $-\frac{b}{a}$, in which case $Q(ax+b)$ will vanish, because one of its factors becomes zero, and R will remain the same, because it does not contain x.

Hence $$f\left(-\frac{b}{a}\right)=R.$$

and therefore the remainder is the value of the dividend when x has the value which makes the divisor vanish.

EXAMPLES.

1. Find the remainder when x^3-3x^2+2x-7 is divided by $x-2$.

Here $$x^3-3x^2+2x-7=Q(x-2)+R,$$

and since this is true for all values of x it will be true when $x=2$, in which case we have

$$2^3-3\times 2^2+2\times 2-7=0+R$$
$$\therefore R=8-12+4-7$$
$$=-7.$$

2. Find the remainder when $8x^3+12x^2-4x-5$ is divided by $2x+3$.

Here $f(x)=8x^3+12x^2-4x-5$, and $x=-\frac{3}{2}$ makes the divisor vanish.

$$\therefore R=f(-\tfrac{3}{2})=8(-\tfrac{3}{2})^3+12(-\tfrac{3}{2})^2-4(-\tfrac{3}{2})-5$$
$$=-27+27+6-5$$
$$=1.$$

3. Find the remainder when $x^n - a^n$ is divided by $x+a$, n being an integer.

Here $R=(-a)^n - a^n$, the first term of which is $-a^n$, or $+a^n$, according as n is odd or even; therefore $R=-2a^n$, or 0, according as n is odd or even.

Thus $x-a$, x^3-a^3, x^5-a^5, x^7-a^7 have remainders $-2a$, $-2a^3$, $-2a^5$, $-2a^7$, respectively, when divided by $x+a$; whilst x^2-a^2, x^4-a^4, x^6-a^6 are exactly divisible by $x+a$.

4. Show that $(b+c-a)(c+a-b)(a+b-c)+8abc$ is exactly divisible by $a+b+c$.

Here the dividend may be denoted by $f(a)$, and since $-b-c$ when substituted for a makes the divisor vanish,

$$\begin{aligned} R=f(-b-c)&=(b+c+b+c)(c-b-c-b)(-b-c+b-c)\\ &\quad+8bc(-b-c)\\ &=2(b+c)(-2b)(-2c)-8bc(b+c)\\ &=8bc(b+c)-8bc(b+c)\\ &=0. \end{aligned}$$

Exercises, XXIII.

Find without actual division the remainder after dividing

1. $5x^4-6x^3+7x-3$ by $x+1$.
2. $6x^3-5x^2+10x-100$ by $x-3$.
3. $18x^2-27x+40$ by $3x-4$.
4. $8x^3-16x^2-12x-10$ by $2x+5$.
5. x^7+a^7 by $x+a$.
6. x^5-a^5 by $x-a$.
7. x^9+a^9 by $x-a$.
8. $x^2-(a+1)x+2a-1$ by $x-a+1$.
9. Prove that $(b-c)a^3+(c-a)b^3+(a-b)c^3$ is exactly divisible by $a+b+c$.
10. Prove that $a^2(b^2+c^2-a^2)+b^2(c^2+a^2-b^2)+c^2(a^2+b^2-c^2)$ is exactly divisible by $a+b+c$.

96. The value of any quantity $f(x)$ when $x=a$ may be determined by finding by actual division the remainder when $f(x)$ is divided by $x-a$.

For the value required is $f(a)$, and this by Art. 95 is the remainder when $f(x)$ is divided by $x-a$.

Thus the value of x^2-7x+9 when $x=5$ is $5^2-7\times5+9$; but the remainder when x^2-7x+9 is divided by $x-5$ is also $5^2-7\times5+9$. Hence the value required may be determined by finding the remainder by actual division.

EXAMPLES.

1. Find the value of x^3-4x+3 when $x=3$.

Here the value required will be the same as the remainder when x^3-4x+3 is divided by $x-3$.

$$\begin{array}{r|r|r} & 1+0-4 & +3 \\ 3 & +3+9 & +15 \\ \hline & 1+3+5 & +18 \end{array}$$

Therefore 18 is the value required.

2. Find the value of $5x^5-4x^4+3x^3-4x^2+x+4$ when $x=-4$.

$$\begin{array}{r|r|r} & 5-\ 4+\ 3-\ \ 4+\ \ \ 1 & +\ \ 4 \\ -4 & -20+96-396+1600 & -6404 \\ \hline & 5-24+99-400+1601 & -6400 \end{array}$$

Therefore -6400 is the value required.

3. Find the value of $x^6-102x^5+100x^4+102x^3-99x^2-201x$ when $x=101$.

$$\begin{array}{r|r|r} & 1-102+100+102-\ 99-201 & +\ \ 0 \\ 101 & +101-101-101+101+202 & +101 \\ \hline & 1-\ \ 1-\ \ 1+\ \ 1+\ \ 2+\ \ 1 & +101 \end{array}$$

Therefore 101 is the value required.

EXERCISES, XXIV.

Find the value of

1. $7x^4-11x^2+x-50$, when $x=2$.

2. $5x^3-2x^2+10x-28$, when $x=4$.

3. x^5-19x^2-121, when $x=-3$.

4. $x^5-98x^4-98x^3-100x^2+98x+100$, when $x=99$.

5. $2x^5+401x^4-199x^3+399x^2-602x+211$, when $x=-201$.

6. $10x^4-1109x^3-109x^2-212x-1111$, when $x=111$.

7. $x^5-4x^3-2x^2$, when $x=-4$.

8. $72x^3-48x$, when $x=-\frac{1}{6}$.

9. $1000x^4-81x$, when $x=0.1$.

97. Although the arrangement of the terms or factors of a quantity is arbitrary and does not affect its value, it will be found useful in some cases to prefer one arrangement to another. Whenever, for example, the parts of an expression are analogous to each other, corresponding letters involved should be arranged in the order of the alphabet, the last of the letters being followed in order by the first.

Thus if a, b, c be the letters, a will be followed by b, b by c, and c by a, in the same manner as if they were arranged on the circumference of a circle.

98. Quantities arranged according to this law are said to be written in *alphabetical circular order*. The following are examples of its application.

Ex. 1. $a+b+c$; $-a-b-c$; $a^2+b^2+c^2$; $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}$.

It will be observed that corresponding parts have like signs.

Ex. 2. $ax+by+cz$; $a^2x+b^2y+c^2z$; $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}$.

Ex 3. $abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=bc+ca+ab$.

In the latter form it will be observed that the first factors of the three terms are b, c, a, and the second factors c, a, b, and that the first term does not contain a, the second does not contain b, and the third does not contain c.

Ex. 4. The quantity $a^2(b-c)-b^2(a-c)-c^2(b-a)$ may be arranged in the more symmetrical form $a^2(b-c)+b^2(c-a)+c^2(a-b)$ in which the corresponding parts follow each other in alphabetical circular order.

Ex. 5. $a(b-c)^2+b(c-a)^2+c(a-b)^2$.

Ex. 6. $(bz-cy)^2+(cx-az)^2+(ay-bx)^2$.

Here the first part does not involve a or x, the second part does not involve b or y, and the third part does contain c or z.

99. When we know the letters involved in a quantity which can be arranged according to the above method, we can infer one part from another, and therefore may write the first part only and leave the rest to be inferred.

Thus the quantities in Exs. 4 and 5 may be written $a^2(b-c)+$&c., and $a(b-c)^2+$ &c., respectively.

Thus also $\frac{ax}{yz}(b^2-c^2)+$&c. may stand for

$$\frac{ax}{yz}(b^2-c^2)+\frac{by}{zx}(c^2-a^2)+\frac{cz}{xy}(a^2-b^2).$$

100. In the same manner as a^n is used to symbolise the product of n a's, $+^n$ and $-^n$ may be employed to denote combinations of n $+$ signs, and n $-$ signs, respectively.

Thus, $(-a)^3=-^3 a^3=-a^3$; $(-x)^2=-^2 x^2=+x^2$; and, generally, if n be a positive integer, so that $2n$ is an even integer and $2n+1$ an odd integer,

$(+x)^{2n}=-^{2n}x^{2n}=+x^{2n}$; $(-x)^{2n+1}=-^{2n+1}x^{2n+1}=-x^{2n+1}$.

Thus also we may write $+^2+^3=+^5=+$; $-^3-^4=-^7=-$; $-^3-^5=-^8=+$; and, generally,

$$+^m+^n=+^{m+n};$$
$$-^m-^n=-^{m+n}.$$

101. By assuming this Law to hold for all values of m and n, positive or negative, integral or fractional, we would be thus enabled to extend our notation for representing the designative elements of quantities by the introduction of such signs as $+^{\frac{1}{2}}$, $-^{\frac{1}{2}}$, &c. We would be thus led to the mode of representing imaginary quantities, the discussion of whose properties is beyond the scope of this Treatise.

102. Also, if the Law hold for zero values of m and n, it will follow that $+^0=+=-^0$.

For $+^0\cdot+^2=+^2=+$; and $+\cdot+^2=+$; therefore $+^0=+$.

Again, $-^0-^3=-^3=-$; and $+-^3=-$; therefore $-^0=+$.

Thus $(+x)^0=+^0\cdot x^0=+1$; $(-x)^0=-^0\cdot x^0=+1$.

INVOLUTION.

103. The process by which the power of a polynomial is expressed as the sum of a series of monomials is called *Involution*, and the power so expressed is said to be *developed*, or *expanded*.

104. In the following Articles we shall give the method of effecting the *development* or *expansion* when the index of the power is 2, 3, 4, -1, or -2.

105. The expansion of the square of any quantity is most easily obtained by the method already given in Art. 72. The two following modes, however, of arranging the results given by that method will be found useful.

106. I. In this method the various parts of the expansion will be written in horizontal rows. In the first row occurs the square of the first term of the given quantity whose square is to be developed. In the second row the product of twice the first term added to the second and the second. In the third row the product of twice the first term added to twice the second term added to the third and the third. And so on.

EXAMPLES.

1. $(a+b)^2= \quad a^2$
$$\qquad +(2a+b)\,b=\&c.$$

2. $(b-c)= \quad b^2$
$$\qquad +(2b-c)\,(-c)=\&c.$$

3. $(a+b+c)^2= \quad a^2$
$$\begin{aligned} &+(2a+b)\,b. \\ &+(2a+2b+c)\,c=\&c. \end{aligned}$$

$$\begin{aligned} 4.\ (a-b-c)^2 = {} & a^2 \\ & +(2a-b)\,(-b) \\ & +(2a-2b-c)\,(-c) = \&c. \end{aligned}$$

$$\begin{aligned} 5.\ (a+b+c+d)^2 = {} & a^2 \\ & +(2a+b)\,b \\ & +(2a+2b+c)\,c. \\ & +(2a+2b+2c+d)\,d = \&c. \end{aligned}$$

From this mode of arrangement is deduced the inverse process for extracting the square roots of quantities.

EXERCISES, XXV.

Develop

1. $(x-y)^2$. 2. $(2x-3y)^2$. 3. $(1-x+y)^3$
4. $(2a-3b-c)^2$. 5. $(yz+zx+xy)$.
6. $(2x-1+x^{-1})^2$. 7. $(x^2+a-ax^{-1})^2$.
8. $(x-y+z-1)^2$. 9. $(a^2-b+c^2-d)^2$.
10. $(2x^2-3x+1-x^{-1})^2$.

107. II. The second method of arranging the development will be evident from the following

EXAMPLES.

1. $(a+b)^2 = a^2+2ab+b^2$.
2. $(a+b+c)^2 = a^2+2a\,(b+c)+b^2+c^2+2bc$.
3. $(a+b+c+d)^2 = a^2+2a\,(b+c+d)+b^2+c^2+d^2+2bc+2bd+2cd$.

108. The development of the cube of a quantity willbe determined by multiplying its square, written according to the second method given above by the quantity itself.

$$\begin{aligned} 1.\ (a+b)^3 = {} & a\,(a+b)^2 \\ & +b\,(b+a)^2 \\ = {} & a^3+2a^2b+ab^2 \\ & +b^3+2b^2a+ba^2 \\ = {} & a^3+3a^2b \\ & +b^3+3b^2a. \end{aligned}$$

Thus ([illegible])

Here the two parts of the development are arranged in horizontal rows, the corresponding parts being placed below each other in alphabetical circular order.

2. $$\begin{aligned}(a+b+c)^3 = {} & a(a+b+c)^2 \\ & +b(b+c+a)^2 \\ & +c(c+a+b)^2 \\ = {} & a^3+2a^2(b+c)+a(b^2+c^2)+2abc \\ & +b^3+2b^2(c+a)+b(c^2+a^2)+2bca \\ & +c^3+2c^2(a+b)+c(a^2+b^2)+2cab \\ = {} & a^3+3a^2(b+c) \\ & +b^3+3b^2(c+a) \\ & +c^3+3c^2(a+b)+6abc.\end{aligned}$$

Here the three parts of the development are arranged in horizontal rows, the corresponding parts being placed below each other in alphabetical circular order as before. In taking the last step, the second and third columns are combined into one, and the terms in the fourth column are added together, giving $6abc$.

3. $$\begin{aligned}(a+b+c+d)^3 = {} & a(a+b+c+d)^2 \\ & +b(b+c+d+a)^2 \\ & +c(c+d+a+b)^2 \\ & +d(d+a+b+c)^2 \\ = {} & a^3+2a^2(b+c+d)+a(b^2+c^2+d^2) \\ & \qquad +2abc+2abd+2acd \\ & +b^3+2b^2(c+d+a)+b(c^2+d^2+a^2) \\ & \qquad +2bcd+2bca+2bda \\ & +c^3+2c^2(d+a+b)+c(d^2+a^2+b^2) \\ & \qquad +2cda+2cdb+2cab \\ & +d^3+2d^2(a+b+c)+d(a^2+b^2+c^2) \\ & \qquad +2dab+2dac+2dbc \\ = {} & a^3+3a^2(b+c+d)+6bcd \\ & +b^3+3b^2(c+d+a)+6cda \\ & +c^3+3c^2(d+a+b)+6dab \\ & +d^3+3d^2(a+b+c)+6abc.\end{aligned}$$

109. From the above examples we can deduce the following rule for the development of the cube of any quantity containing three or four terms.

The cube of any quantity is equal to the sum of the cube of each term, and three times the square of each term multiplied by the sum of the other terms, added to six times the sum of the products of every three terms.

The above Rule, although deduced for quantities containing not more than four terms, is nevertheless true for the development of the cube of a quantity containing more terms than four.

110. The cube of a binomial may also be put in the following convenient form:

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b),\quad (a-b)^3 = a^3 - b^3 - 3ab(a-b).$$

Exercises, XXVI.

Develop

1. $(y+z)^3$. 2. $(2a-3b)^3$ 3. $(2a^2+3)^3$
4. $(x^2-2)^3$. 5. $(2x+3y^2)^3$. 6. $(1-a+a^2)^3$.
7. $(x-1+x^{-1})^3$. 8. $(x^2+2-x^{-2})^3$.
9. $(1-x+y-z)^3$. 10. $(bc+ca+ab)^3$.

111. The results given by the Rule of Art. 109 may also be arranged according to the method of the following examples, in which the various parts are arranged in horizontal rows.

1. $(a+b)^3 = \quad a^3$
$\qquad\qquad +(3a^2+3ab+b^2)\,b = \&c.$

2. $(a+b+c)^3 = \quad a^3$
$\qquad\qquad +(3a^2+3ab+b^2)\,b$
$\qquad\qquad +(3\,\overline{a+b}^{\,2} + 3\,\overline{a+b}\,.\,c + c^2)\,c = \&c.$

3. $$\begin{aligned}(a+b+c+d)^3 = \; & a^3 \\ & +(3a^2+3ab+b^2)\,b \\ & +(3\,\overline{a+b}^2+3\,\overline{a+b}\,.\,c+c^2)\,c \\ & +(3\,\overline{a+b+c}^2+3\,\overline{a+b+c}\,.\,d+d^2)\,d \\ & = \&c.\end{aligned}$$

4. $$\begin{aligned}(2x^2-x+3)^3 = \; & (2x^2)^3 \\ & +(12x^4-6_{x3}+x^2)\,(-x) \\ & +(3\,.\,2\overline{x^2-x}^2+9\,.\,2\overline{x^2-x}+9)(3) = \&c.\end{aligned}$$

From this mode of arrangement is deduced the inverse process for extracting the cube root of a quantity.

Exercises, XXVII.

Develop

1. $(x+y)^3$ 2. $(x-y)^3$ 3. $(a-b^2)^3$.
4. $(x-y-z)^3$. 5. $(x^2+y-z^2)^3$.
6. $(1+x-x^2)^3$.

112. The fourth power of a polynomial is found by squaring its square.

Thus $(1+x)^4 = \{(1+x)^2\}^2 = \{1+2x+x^2\}^2$
$= 1+4x+6x^2+4x^3+x^4$.

Exercises, XXVIII.

Develop

1. $(1-a)^4$. 2. $(x-y)^4$. 3. $(1-x+x^2)^4$.
4. $(a^2-b+c)^4$.

113. The development of the negative integral powers -1, -2 of polynomials may be effected by expressing them as the reciprocals of the corresponding positive powers.

Examples.

1. Develop $(1+x)^{-1}$.

Here $(1+x)^{-1} = \frac{1}{1+x}$, and if 1 be divided by $1+x$ the quotient will be $1-x+x^2-x^3+$&c., and the remainder will depend on the stage at which we stop dividing. The quotient $1-x+x^2-x^3+$&c. is then the development of $(1+x)^{-1}$; but it must be carefully observed that such developments do not express the exact values of the powers unless the remainders be taken into account. Thus in the present example if we stop at $-x^3$ in the quotient, we shall have

$$(1+x)^{-1}=1-x+x^2-x^3+\frac{x^4}{1+x}.$$

If we stop at $+x^4$ in the quotient, we shall have

$$(1+x)^{-1}=1-x+x^2-x^3+x^4-\frac{x^5}{1+x},$$

and so on.

2. Develop $(1-x)^{-2}$.

Here $(1-x)^{-2}=\dfrac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+$&c. by actual division. If we stop at $4x^3$ in the development, the remainder will be $5x^4-4x^5$.

Exercises, XXIX.

Develop to four terms

1. $(1-x)^{-1}$. 2. $(1+x^2)^{-1}$. 3. $(1+x+x^2)^{-1}$.

4. $\dfrac{1+x^3}{1-x}$. 5. $\dfrac{1+2x+3x^2}{1+x}$. 6. $\dfrac{1+x+x^2}{(1+x)^2}$.

EVOLUTION.

114. The process by which the root of a polynomial is expressed as the sum of a series of mononomials is called *Evolution*, and the root when so expressed is said to be *extracted*.

115. In the case of the square and cube roots the extraction can be effected by processes the inverse

of those given in Involution for the development of the squares and cubes of polynomials.

We shall consider in the first instance the method of extracting the square-root of a quantity.

116. Since the square of a quantity is equal to the square of the same quantity with its sign or signs changed, it will follow that there will be two square roots, one being derived from the other by a change of signs.

Thus since $(+a)^2=(-a)^2=a^2$, it follows that the square root of a^2 is $+a$, or $-a$. These two values may be represented by the symbol $\pm a$, the two signs being combined.

Again, since $(a-b)^2=(-a+b)^2=a^2-2ab+b^2$, it follows that the square root of $a^2-2ab+b^2$ is $a-b$, or $-a+b$.

In the following articles we shall only determine the root whose first term is positive.

117. In the process of Art. 106, the successive terms of the quantity to be squared occur as factors of the several products written in the horizontal rows, and therefore, in extracting the square root of such a development, the successive terms of the square root may be found by arranging the given development in the form of such successive products.

$$\text{Thus } a^2+2ab+b^2=a^2$$

$$+(2a+b)b$$

from which arrangement we infer that the square root of $a^2+2ab+b^2$ is $a+b$.

Ex. 2.
$$\begin{aligned} 4-12x^2+9x^4 &= 2^2 \\ &\quad +(4-3x^2)\,(-3x^2). \end{aligned}$$

Therefore the square root of $4-12x^2+9x^4$ is $2-3x^2$.

Ex. 3.
$$\begin{aligned} a^2-2ab+b^2-2ac+2bc+c^2 &= a^2 \\ &\quad +(2a-b)\,(-b) \\ &\quad +(2a-2b-c)\,(-c) \end{aligned}$$

Therefore $a-b-c$ is the square root of a^2-2ab+ &c.

118. The preceding process may be more conveniently replaced by the following equivalent one in which the successive terms of the square root are written as they are obtained to the right of the quantity whose square root is required.

Ex. 1.
$$\begin{array}{r|l} & a^2+2ab+b^2\ (a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & \underline{2ab+b^2} \end{array}$$

Here the first term a is the square root of the first term of the given quantity from which its square a^2 is subtracted leaving $2ab+b^2$. The second term b is obtained by dividing the first term $2ab$ of this remainder by $2a$, double the term already found; b is then added to $2a$ forming the complete divisor $2a+b$, and this multiplied by b gives $2ab+b^2$

Ex. 2.
$$\begin{array}{r|l} & 4-12x^2+9x^4\ (2-3x^2 \\ & 4 \\ \hline 4-3x^2) & -12x^2+9x^4 \\ & \underline{-12x^2+9x} \end{array}$$

Ex. 3.

$$\begin{array}{rl}
 & a^2-2ab+b^2-2ac+2bc+c^2\;(a-b-c \\
 & a^2 \\
2a-b) & \overline{-2ab+b^2} \\
 & -2ab+b^2 \\
2a-2b-c) & \overline{-2ac+2bc+c^2} \\
 & \underline{-2ac+2bc+c^2}
\end{array}$$

Here the first two terms a and $-b$ are found as before. The third term $-c$ is obtained by dividing $-2ac$ by $2a$, the second divisor being formed by doubling $a-b$, the part of the root already found, and adding to it $-c$. The third term $-c$ is then multiplied into the second divisor $2a-2b-c$ giving the product $-2ac+2bc+c^2$.

In the same manner a third divisor would be formed by doubling the three terms of the square root previously found and adding the fourth term; and so on.

It will be observed that the successive terms of the root and the successive divisors correspond to the factors of the development of the square, when arranged according to the method of Art. 106.

Ex. 4.

$$\begin{array}{rl}
 & x^4-6x^3+13x^2-12x+4\;(x^2-3x+2 \\
 & x^4 \\
2x^2-3x) & \overline{-6x^3+13x^2} \\
 & -6x^3+9x^2 \\
2x^2-6x+2) & \overline{4x^2-12x+4} \\
 & \underline{4x^2-12x+4}
\end{array}$$

Ex. 5.

$$\begin{array}{rl}
 & \tfrac{1}{4}x^4-x^{\frac{5}{2}}-x+4x^{-\frac{1}{2}}+4x^{-2}\;(\tfrac{1}{2}x^2-x^{\frac{1}{2}}-2x^{-1} \\
 & \tfrac{1}{4}x^4 \\
x^2-x^{\frac{1}{2}}) & \overline{-x^{\frac{5}{2}}-x} \\
 & -x^{\frac{5}{2}}+x \\
x^2-2x^{\frac{1}{2}}-2x^{-1}) & \overline{-2x+4x^{-\frac{1}{2}}+4x^{-2}} \\
 & \underline{-2x+4x^{-\frac{1}{2}}+4x^{-2}}
\end{array}$$

119. In the foregoing examples the process of extraction terminates, and the quantity whose root is extracted is said to be a *perfect* square. When the same process is applied to a quantity which is not a perfect square, a result is obtained, the square of which, added to the last difference, is equal to the given quantity.

For example, let the process be applied to find the square root of $1+x$.

$$\begin{array}{r|l}
 & 1+x\,(1+\frac{1}{2}x-\frac{1}{8}x^2 \\
 & 1 \\ \cline{2-2}
2+\frac{1}{2}x) & x \\
 & x+\frac{1}{4}x^2 \\ \cline{2-2}
2+x-\frac{1}{8}x^2) & -\frac{1}{4}x^2 \\
 & -\frac{1}{4}x^2-\frac{1}{8}x^3+\frac{1}{64}x^4 \\ \cline{2-2}
 & \frac{1}{8}x^3-\frac{1}{64}x^4
\end{array}$$

If the process be stopped at this stage, it will follow that

$$(1+\tfrac{1}{2}x-\tfrac{1}{8}x^2)^2+\tfrac{1}{8}x^3-\tfrac{1}{64}x^4=1+x.$$

120. Care must be taken to arrange the given quantity according to ascending or descending powers of some one letter; otherwise the process may not terminate, although the quantity be a perfect square.

For example, in finding the square root of $4x^4+4x^2+1$ let the quantity be arranged in the order $4x^2+4x^4+1$.

$$\begin{array}{r|l}
 & 4x^2+4x^4+1\,(2x+x^3 \\
 & 4x^2 \\ \cline{2-2}
4x+x^3) & 4x^4+1 \\
 & 4x^4+x^6 \\ \cline{2-2}
 & -x^6+1
\end{array}$$

By proceeding in this manner we would get a series of terms in the root, the square of which added to the last difference would be equal to $4x^4+4x^2+1$. Thus if we stop at the second term of the root we have

$$(2x+x^3)^2-x^6+1=4x^4+4x^2+1.$$

EXERCISES, XXX.

Extract the square root of

1. $4x^4+4ax^2+a^2$.
2. $4a^2-12ab+9b^2$.
3. $x^2-px+\frac{p^2}{4}$.
4. $a^4+2a^3+3a^2+2a+1$.
5. $\frac{9}{4}a^4+2a^2b^2+\frac{4}{9}b^4$.
6. $4x^{-4}+12x^{-3}+9x^{-2}$.
7. $4x-4x^{\frac{1}{2}}+1$.
8. $x^2+4x+2-4x^{-1}+x^{-2}$.
9. $9a^2-12ab+24ac-16bc+4b^2+16c^2$.
10. $36x^4-36x^3+17x^2-4x+\frac{4}{9}$.
11. $x^4+8x^2+24+32x^{-2}+16x^{-4}$.
12. $x^{2n}-2px^n+p^2$.
13. $x-2+2x^{-\frac{1}{2}}+x^{-1}-2x^{-\frac{3}{2}}+x^{-2}$.
14. $x^{2n}+2px^{\frac{3n}{2}}+(p^2+2q)\,x^n+2pqx^{\frac{n}{2}}+q^2$.
15. $25a^8+20a^5b^3-30a^4b^4+4a^2b^6-12ab^7+9b^8$.

121. We now proceed to the method of extracting the cube root of a quantity.

The process of extracting the cube root of a quantity is the inverse of that given in Art. 111 for the development of the cube, and may be conducted as in the following examples, in which the several terms of the root after the first are obtained by dividing the first terms of the successive remainders by the first terms of the several divisors.

Ex. 1.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3\,(a+b \\
 & a^3 \\ \cline{2-2}
3a^2+3ab+b^2\,) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\ \cline{2-2}
\end{array}$$

Here the first term a of the root is the cube root of a^3. The second term b is the quotient of $3a^2b$ by $3a^2$, the first term of the divisor being three times the square of the first term of the root. The second term of the divisor is three times the product of the first and second terms of the root, and the third term of the divisor is the square of the second term of the root.

Ex. 2.

$$\begin{array}{r|l}
 & x^3-9x^2+27x-27\,(x-3 \\
 & x^3 \\ \cline{2-2}
3x^2-9x+9\,) & -9x^2+27x-27 \\
 & -9x^2+27x-27. \\ \cline{2-2}
\end{array}$$

Ex. 3.

$$\begin{array}{r|l}
 & x^6-3x^5-3x^4+11x^3+6x^2-12x-8\,(x^2-x-2 \\
 & x^6 \\ \cline{2-2}
3x^4-3x^3+x^2\,) & -3x^5-3x^4+11x^3 \\
 & -3x^5+3x^4-x^3 \\ \cline{2-2}
3x^4-6x^3-3x^2+6x+4\,) & -6x^4+12x^3+6x^2-12x-8 \\
 & -6x^4+12x^3+6x^2-12x-8. \\ \cline{2-2}
\end{array}$$

The second divisor is equal to $3(x^2-x)^2+3(x^2-x)(-2)+(-2)^2$, the first part of which is three times the square of the part of the root previously found. The first term $3x^4$ is divided into $-6x^4$ giving the quotient -2, the third term of the root, and the remaining parts of the second divisor $3(x^2-x)(-2)$ and $(-2)^2$ are then formed, and the divisor is complete.

122. When there are four terms in the root the third divisor is formed by adding together three times the square of the sum of the first three terms of the root, three times the product of that sum and the fourth term, and the square of the fourth term of the root.

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x-2x^{\frac{2}{3}}+3} \\ \text{Ex. 4. } x^3-6x^{\frac{8}{3}}+12x^{\frac{7}{3}}+x^2-36x^{\frac{5}{3}}+36x^{\frac{4}{3}}+27x-54x^{\frac{2}{3}}+27 \\ \qquad\quad \underline{x^3 \qquad\qquad\qquad\qquad\qquad} \\ 3x^2-6x^{\frac{5}{3}}+4x^{\frac{4}{3}})-6x^{\frac{8}{3}}+12x^{\frac{7}{3}}+x^2 \\ \qquad\qquad\qquad\quad -6x^{\frac{8}{3}}+12x^{\frac{7}{3}}-8x^2 \\ 3x^2-12x^{\frac{5}{3}}+12x^{\frac{4}{3}}+9x-18x^{\frac{2}{3}}+9)9x^2-36x^{\frac{5}{3}}+36x^{\frac{4}{3}}+27x- \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 54x^{\frac{2}{3}}+27 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 9x^2-36x^{\frac{5}{3}}+36^{\frac{4}{3}}+27x- \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 54x^{\frac{2}{3}}+27. \end{array}$$

It will be observed that the successive terms of the root and the successive divisors correspond to the factors of the development of the cube, when arranged according to the method of Art. 111.

EXERCISES, XXXI.

Extract the cube root of

1. $1+6x+12x^2+8x^3$. 2. $27x^3-54x^2+36x-8$.
3. $8a^3-84a^2x+294ax^2-343x^3$.
4. $27x^3+54x+36x^{-1}+8x^{-3}$.
5. $8x^6-12x^5+18x^4-13x^3+9x^2-3x+1$.
6. $27x^6-54ax^5+63a^2x^4-44a^3x^3+21a^4x^2-6a^5x+a^6$.

THE GREATEST COMMON MEASURE.

123. A quantity is said to be of so many *dimensions*, in any letter, as are indicated by the index of the highest power of that letter involved in it.

Thus x^2y is of 2 dimensions in x, and of 1 dimension in y; x^3-2x+3 is of 3 dimensions in x; $x^{\frac{3}{2}}-x-1$ is of $\frac{3}{2}$ dimensions in x; $x^{-1}y^2+a^2x^{-2}y$ is of -1 dimension in x, and of 2 dimensions in y.

124. By the phrase *total dimensions* is meant the

greatest sum of the indices of the powers involved in any one term.

Thus the total dimensions of x^3y^2+3xy are 5, and the the total dimensions of $x^{-1}y^3+x^{-2}y$ are 2.

125. When the total dimensions of each term of a quantity are the same, the quantity is said to be *homogeneous.*

Thus the quantities $x^2-xy+3y^2$, $x^4+3x^3y+x^2y^2-5xy^3+y^4$, are homogeneous.

126. Any quantity can be rendered homogeneous in form by introducing proper powers of some symbol, as y, whose numerical value is 1, as factors of the several terms.

Thus x^2-3x+2 and x^3-3x^2+3x-1 may be written in the homogeneous forms $x^2-3xy+2y^2$ and $x^3-3x^2y+3xy^2-y^3$, if $y=1$.

127. The dimensions of any term of a quantity are said to be the same as the dimensions of the quantity.

Thus, though the dimensions of $-2x^2$, considered by itself, are 2, yet as a part of the quantity x^4-2x^2+3 its dimensions are 4, as also are the dimensions of the third term $+3$. So also the dimensions of every term of $2x^3-x^2+2x-1$ are 3.

128. This language is adopted in the use of the word dimensions, in order that our Algebraical phraseology may accord with that of Geometry in those cases where Algebraical symbols are taken to represent Geometrical magnitudes.

Thus the quantity $2x+5$, if x be taken to represent a line, will have no meaning unless $+5$ also represents a line, that is, unless the terms $2x$ and $+5$ have the same Geometrical dimension. Again, the quantity x^2+5x-3, if x^2 be taken to represent an area, will have no meaning unless $+5x$ and -3 also represent areas, that is, unless the terms $+5x$ and -3 have the same Geometrical dimensions as x^2; and the quantity x^3-3x+2, if x^3 be taken to represent a volume, will have no meaning unless $-3x$ and $+2$ also represent volumes, that is, unless the terms $-3x$ and $+2$ have the same Geometrical dimensions as x^3, namely, length, breadth, and thickness. The preceding phraseology where the dimensions are 1, 2, or 3 is then extended to quantities of any dimensions whatsoever.

129. A *whole expression* or *quantity* is one which involves powers with positive indices (including zero), and integral coefficients only.

Thus $-5x^3$, x^2-2x+3, $6x^3-5x^{\frac{3}{2}}+7x-10x^{\frac{1}{2}}+9$ are whole expressions, as are also all positive and negative integers, since they may be considered as coefficients of the zero power of any symbol.

130. When two or more whole expressions are multiplied together, each is said to be a *measure* of the product, and the product is said to be a *multiple* of each factor.

Thus 3, x, and $x+1$ are measures of $3x^2+3x$; 5, x^2, $x-1$, $y+1$, and $y-1$ are measures of $5x^2(x-1)(y^2-1)$; and these products are multiples of each of their respective factors.

131. A quantity, therefore, cannot be considered a measure or a multiple of another unless it be a whole expression.

132. Also to get a multiple or measure of a quantity, we must multiply or divide it by some whole

expression, and the product or quotient so obtained must also be a whole expression.

133. The *greatest* measure of a quantity is the quantity divided by $+1$, or -1, that divisor being taken which will make the first term of the quotient positive.

Thus the greatest measure of $5x-3$ is $5x-3$, and of $-2x^2+x-3$ is $2x^2-x+3$.

134. The *least* multiple of a quantity is the quantity multiplied by $+1$, or -1, that multiplier being taken which will make the first term of the product positive.

Thus the least multiple of $x-4$ is $x-4$, and of $-5x^2-2x+1$ is $5x^2+2x-1$.

135. When one quantity is a measure of two others it will measure the sum and difference of any multiples of these.

To prove this proposition, let the quantities be denoted by f_1, f_2, f_3; and let f_1 measure f_2 and f_3, so that $f_2=af_1$ and $f_3=bf_1$, where a and b are whole expressions.

Take any multiples mf_2, nf_3, of f_2, f_3, m and n being any whole expressions whatsoever. Then since $mf_2=maf_1$, and $nf_3=nbf_1$, we have

$$mf_2 \pm nf = maf_1 \pm nbf_1 = (ma \pm nb)\, f_1.$$

Therefore f_1 is a measure of $mf_2 \pm nf_3$, because $ma \pm nb$ is a whole expression; and the proposition is proved.

EXAMPLES.

1. $-2x^2$ which is a measure of $6x^3$ and $-8x^2y$ will measure the sum of any multiples of these, as $6x^3(-2xy)$ $-8x^2y\,(x-1)$.

For this sum $= -12x^4y - 8x^2y\,(x-1)$
$= -2x^2\{6x^2y + 4xy - 4y\}$.

Hence $-2x^2$ is a measure of the sum because the factor $6x^2y + 4xy - 4y$ is a whole expression.

2. $x-1$ is a measure of x^2-1 and x^3-1, and will therefore measure $(x^2-1)\,(-3x) - (x^3-1)\,(x+1)$.

For this difference

$$=(x-1)\,\{(x+1)\,(-3x) - (x^2+x+1)\,(x+1)\}$$
$$=(x-1)\,(x+1)\,\{-x^2-4x-1\}$$

Hence $x-1$ is a measure of this difference, because $(x+1)\,(-x^2-4x-1)$ is a whole expression.

136. When one quantity is a measure of two or more others, it is said to be a *common* measure of those quantities.

Thus $2x$ is a common measure of $4x^3$ and $2x^2+4x$.

137. The *greatest common measure* of two or more quantities is the common measure of highest dimensions and greatest numerical coefficient or coefficients.

Thus 2, x, $2x$ and $2x^2$ are common measures of $4x^3$ and $6x^2y$, but $2x^2$ is their greatest common measure.

138. We shall consider the process for finding the G. C. M. in the three following cases, namely, I. When one of the quantities is a mononomial. II. When the two quantities are polynomials, neither of which has a simple factor. III. When the two quantities are polynomials, one or both of which have simple factors.

139. I. The G. C. M. of two mononomials will be obtained by multiplying the G. C. M. of the numerical coefficients by the least power or powers of the different symbols that occur in both the quantities.

Thus the G. C. M. of $12x$ and $16x^2$ is $4x$; of $10x^2yz$ and $15xy^2z^3$ is $5xyz$; and of $2a^{\frac{1}{2}}bc^2$ and $8ab^{\frac{1}{4}}c^{\frac{1}{3}}$ is $2a^{\frac{1}{2}}b^{\frac{1}{4}}c^{\frac{1}{3}}$.

140. The G. C. M. of a mononomial and a polynomial will be the G. C. M. of the former and the mononomial of greatest numerical coefficient and dimensions contained in the latter.

Thus the G. C. M. of $6x^2yz$ and $9x^2y^2z-9xy^3z$, or $9xy^2(x-y)$, is the G. C. M. of $6x^2yz$ and $9xy^2z$, that is, $3xyz$.

Exercises, XXXII.

Find the G. C. M. of

1. $12ax^2$ and $16a^2x^2$.
2. $9ax^2y^2$ and $15a^2xz$.
3. $10x^2y^3z^4$ and $15x^3y^4z^2$.
4. ab^2cu^2v and $3a^2buv^2w$.
5. $4ab^2$ and $12a^2bx-8ab^2y$.
6. $3x^2y^3u^3v^2$ and $2u^2vx^2y-6uv^2xy^2$.

141. The G. C. M. of two polynomials involving powers of some one letter neither of which contains a simple factor, is obtained by a succession of steps similar to the following, in which a multiple of the G. C. M. is found of less dimensions than either of the given quantities.

Let A and B be the given polynomials, the dimensions of A being not greater than the dimensions of B.

Multiply B, if necessary, by a number a in order to make its first term a multiple of that of A, and proceed as follows:

$$\begin{array}{l} \mathrm{A})\ a\mathrm{B}\ (b \\ \phantom{\mathrm{A})\ }\underline{b\mathrm{A}} \\ \phantom{\mathrm{A})\ }\ \mathrm{C} \end{array}$$

Here b is the first term of the quotient, and C, the first difference, is of less dimensions than B.

Now C being equal to $a\mathrm{B}-b\mathrm{A}$, or the difference of two multiples of A and B, is a multiple of all the common measures of A and B, and therefore of their G. C. M.

Also every common measure of C and A is a measure of $\mathrm{C}+b\mathrm{A}$, or $a\mathrm{B}$, and therefore of B, because A has no simple factor.

Hence the G. C. M. of A and B is the same as the G. C. M. of A and C. Also if $\mathrm{C}=c\mathrm{D}$. where c is the product of all the simple factors in C, the G. C. M. of A and D is the same as the G. C. M. of A and C, and therefore of A and B.

Hence the problem is reduced to finding the G. C. M. of A and D.

These two quantities A and D are then treated in precisely the same manner as A and B, and the process is continued until it terminates as follows, when the last divisor, P (suppose), is a measure of the last dividend, Q.

$$\begin{array}{l} \mathrm{P})\ \mathrm{Q}\ (r \\ \phantom{\mathrm{P})\ }\underline{r\mathrm{P}} \\ \phantom{\mathrm{P})\ }\ \ 0 \end{array}$$

The problem is thus finally reduced to finding the G. C. M. of P and Q, or $r\mathrm{P}$. This is evidently P itself.

Hence the last polynomial divisor in the above process will be the G. C. M. required.

142. It will be noticed in the preceding process that mononomial factors which are also whole expressions are introduced only when *necessary*, and suppressed when *possible*.

The manner in which the process is conducted will be exhibited in the following

EXAMPLES.

1. Find the G. C. M. of x^2+2x-3 and x^2+5x+6.

$$\begin{array}{l}
x^2+2x-3)\,x^2+5x+6\,(1\\
\qquad\qquad\quad \underline{x^2+2x-3}\\
\qquad\qquad\quad \underline{3)\;\;3x+9\quad}\\
\qquad\qquad\qquad\quad x+3)\,x^2+2x-3\,(x-1\\
\qquad\qquad\qquad\qquad\qquad \underline{x^2+3x\quad}\\
\qquad\qquad\qquad\qquad\qquad\;\; -\;x-3\\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{-\;x-3.}
\end{array}$$

Here $3x+9$ is the difference of the given quantities, and therefore a multiple of their G. C. M. Also since $3x+9$ contains 3 as a factor, its other factor $x+3$ must be a multiple of the G. C. M., which contains no simple factor because the given quantities do not. The G. C. M. will therefore be the same as the G. C. M. of $x+3$ and x^2+2x-3; and as the former is a measure of the latter the G. C. M. will be $x+3$ itself.

2. Find the G. C. M. of x^2+2x+1 and x^3+2x^2+2x+1.

$$\begin{array}{l}
x^2+2x+1)\,x^3+2x^2+2x+1\,(x\\
\qquad\qquad\quad \underline{x^3+2x^2+x\qquad\quad}\\
\qquad\qquad\qquad\qquad x\;+1)\,x^2+2x+1\,(x+1\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2+x\qquad\;}\\
\qquad\qquad\qquad\qquad\qquad\qquad\;\; x\;+1\\
\qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{x\;+1}
\end{array}$$

Here $x+1$ is the difference of the second quantity, and x times the first; and is therefore a multiple of their G. C. M. The G. C. M. of the given quantities is therefore the G. C. M. of $x+1$ and x^2+2x+1, that is, $x+1$.

3. Find the G.C.M. of $2x^3-7x-2$ and $2x^2-x-6$.

$$\begin{array}{rll}
2x^2-x-6) & 2x^3-7x-2\,(x,\ +1 & \\
 & \underline{2x^3-x^2-6x} & \\
 & \quad x^2-x-2 & \\
 & \qquad\quad \underline{2} & \\
 & 2x^2-2x-4 & \\
 & \underline{2x^2-x-6} & \\
-1) & \underline{-x+2} & \\
 & x-2)\,2x^2-x-6\,(2x+3 & \\
 & \qquad\quad \underline{2x^2-4x} & \\
 & \qquad\qquad\quad 3x-6 & \\
 & \qquad\qquad\quad \underline{3x-6} &
\end{array}$$

G. C. M. $=x-2$.

Here we first find x^2-x-2 a multiple of the G. C. M. of less dimensions than $2x^3-7x-2$; so that the G. C. M. required is the G. C. M. of $2x^2-x-6$ and x^2-x-2, which is the same as the G. C. M. of $2x^2-x-6$ and $2(x^2-x-2)$. The partial quotients x, $+1$ of $2x^3-7x-2$ and $2x^2-2x-4$ divided by $2x^2-x-6$ are separated by a comma to distinguish them from parts of an ordinary quotient. At the next step, we find $-x+2$ a multiple of the G. C. M. of less dimensions than $2x^2-2x-4$; so that the G. C. M required is the G. C. M. of $2x^2-x-6$ and $-x+2$, which is the same as the G. C. M. of $2x^2-x-6$ and $x-2$, that is, $x-2$, because $2x^2-x-6$ is a multiple of $x-2$.

4. Find the G. C. M. of $7x^2-23x+6$ and $5x^3-18x^2+11x-6$.

$$5x^3-\ 18x^2+11x-6$$
$$7$$
$$7x^2-23x+6)35x^3-126x^2+77x-42(5x,\ 11$$
$$35x^3-115x^2+30x$$
$$-11x\ \ +47x-42$$
$$-\ 7$$
$$77x^2-329x+294$$
$$77x^2-253x+\ 66$$
$$-76)-\ 76x+228$$
$$x-3)7x^2-23x+6(7x-2$$
$$7x^2-21x$$
$$-2x+6$$
$$-2x+6$$

G. C. M. $=x-3$.

Here the second given quantity is multiplied by 7 in order to make its first term a multiple of the first term of $7x^2-23x+6$, and the process is conducted as before.

5. Find the G.C.M. of $14x^4-35x^3+4x^2-12x+5$ and $4x^3-10x^2+6x-15$.

$$14x^4-35x^3+4x^2-12x+\ 5$$
$$2$$
$$4x^3-10x^2+6x-15)28x^4-70x^3+\ 8x^2-\ 24x+10(7x$$
$$28x^4-70x^3+42x^2-105x$$
$$-1)-34x^2+\ 81x+10$$
$$34x^2-\ 81x-10$$

$$
\begin{array}{rl}
 & 4x^3-\ 10x^2+\quad 6x-\ 15 \\
 & 17 \\ \cline{2-2}
34x^2-81x-10) & 68x^3-170x^2+\ 102x-255(2x,\ 4 \\
 & 68x^3-162x^2-\quad 20x \\ \cline{2-2}
 & -\ \ 8x^2+\ 122x-\ 255 \\
 & -\ \ 17 \\ \cline{2-2}
 & 136x^3-2074x+4335 \\
 & 136x^2-\ 324x-\quad 40 \\ \cline{2-2}
-875) & -1750x+4375 \\ \cline{2-2}
2x-5) & 34x^2-81x-10(17x+2 \\
 & 34x^2-85x \\ \cline{2-2}
 & 4x-10 \\
 & 4x-10 \\ \cline{2-2}
\end{array}
$$

G. C. M. $=2x-5$.

Here $4x^3-10x^2+6x-15$ is multiplied in the second step by 17, in order to make its first term a multiple of the first term of $34x^2-81x-10$.

143. The process of the preceding examples will frequently enable us to find the G.C.M. of polynomials involving powers of different letters, as in the following

EXAMPLE.

Find the G. C. M. of $2x^2+xy-3y^2$ and $3x^2-4xy+y^2$.

$$
\begin{array}{rl}
 & 3x^2-4xy\quad +\ y^2 \\
 & 2 \\ \cline{2-2}
2x^2+xy-3y^2) & 6x^2-\ 8xy+\ 2y^2(3 \\
 & 6x^2+\ 3xy-\ 9y^2 \\ \cline{2-2}
-11y) & -11xy+11y^2 \\ \cline{2-2}
x-y) & 2x^2+xy-3y^2(2x+3y \\
 & 2x^2-2xy \\ \cline{2-2}
 & 3xy-3y^2 \\
 & 3xy-3y^2 \\ \cline{2-2}
\end{array}
$$

G. C. M. $=x-y$.

Here the suppression of the factor $-11y$ cannot affect the G. C. M., which contains no simple factor.

144. When the preceding method fails we must introduce, when necessary, and suppress, when possible, polynomial factors, provided they are not common to the given quantities. The reasoning is analogous to that of Art. 141.

EXAMPLE.

Find the G. C. M. of

$2ax^2-(a-2)x-1$ and $2bx^2-(b+2)x+1$.

$$\begin{array}{rl}
 & \underline{\qquad\qquad\qquad\qquad\qquad\qquad a} \\
2ax^2-(a-2)x-1\,) & 2abx^2-a(b+2)x+a\,(b \\
 & \underline{2abx^2-b(a-2)x-b} \\
-(a+b)\,) & \underline{-2(a+b)x+(a+b)} \\
 & 2x-1\,)\,2ax^2-(a-2)x-1\,(ax+1 \\
 & \qquad\qquad \underline{2ax^2-ax} \\
\text{G. C. M.}=2x-1. & \qquad\qquad\qquad 2x-1 \\
 & \qquad\qquad\qquad \underline{2x-1}
\end{array}$$

Here the suppression of the factor $-(a+b)$ which is evidently not a measure of either of the given quantities, will not affect their G. C. M.

EXERCISES, XXXIII.

Find the G. C. M. of

1. x^2-1 and x^2-x-2.
2. x^2+2x-3 and x^2+5x+6.
3. x^2-1 and x^3-x^2+x-1.
4. $x^3-6x^2+11x-6$ and x^3+4x^2+x-6.
5. $2x^2+x-3$ and $3x^2-4x+1$.
6. x^4-1 and x^3+x^2-x-1.
7. $3x^2-22x+32$ and $x^3-11x^2+32x-28$.
8. $2x^2+23x+45$ and $x^{\frac{3}{2}}-3x+9x^{\frac{1}{2}}-27$.

9. $x^4-9x^3+29x^2-39x+18$ and $4x^3-27x^2+56x-33$.
10. $3x^2-38x+119$ and $x^3-19x^2+119x-245$.
11. $9x^3+53x^2-9x-18$ and $x^2+11x+30$.
12. $2x^3+x^2-8x+5$ and $7x^2-12x+5$.
13. $x^4-2x^3+4x^2-6x+3$ and $x^4-x^3-2x^2+3x-1$.
14. $20x^4+x^2-1$ and $25x^4+5x^3-x-1$.
15. $2x^3+9x^2+4x-15$ and $4x^3+8x^2+3x+20$.
16. x^5+5x^3+6 and $x^4+40x+39$.
17. $3x^5-20x^2+15x+2$ and x^4-4x+3.
18. x^4-a^4 and $x^3-bx^2-a^2x+a^2b$.
19. $x^3+x^2y+xy+y^2$ and x^4-y^2.
20. $x^8+x^6y^2+x^2y+y^3$ and x^4-y^4.
21. $5x^2+26xy+33y^2$ and $7x^2+19xy-6y^2$.
22. $3x^4-x^2y^2-2y^4$ and $2x^3+3x^2y-2xy^2-3y^3$.

145. III. The G. C. M. of two polynomials, involving simple factors, is the product of the G. C. M. of the simple factors and the G. C. M. of the polynomial factors.

EXAMPLES.

1. Find the G. C. M. of $8x^2+12x+4$ and $6x^2-6x-12$.

Here the given quantities are equivalent to $4\ (2x^2+3x+1)$ and $6\ (x^2-x-2)$. The G. C. M. of 4 and 6 is 2, and the G. C. M. of $2x^2+3x+1$ and x^2-x-2 is $x+1$.

Therefore the G. C. M. required is $2\ (x+1)$.

2. Find the G. C. M. of $20x^4y+104x^3y^2+132x^2y^3$ and $42x^3y^2+114x^2y^3-36xy^4$.

The first quantity is equal to $4x^2y\ (5x^2+26xy+33y^2)$ and the second quantity to $6xy^2\ (7x^2+19xy-6y^2)$.

The G. C. M. of the simple factors is $2xy$, and of the compound factors is $x+3y$.

Therefore the G. C. M. required is $2xy\,(x+3y)$.

EXERCISES, XXXIV.

Find the G. C. M. of

1. $24x^2-40x+16$ and $24x^3-24x^2-6x+6$.

2. $6x^4y+12x^3y^2+12x^2y^3+6xy^4$ and $9x^5+9x^3y^2+9xy^4$.

3. $4a^2x^3+2a^2x^2y-6a^2xy^2$ and $9ax^5-3ax^4y-3ax^3y^2-3ax^2y^3$.

4. $2a^6b^2+3a^5b^3-3a^4b^4-2a^3b^5+a^2b^6-ab^7$ and $2a^7b+3a^6b^2+a^5b^3+4a^4b^4-a^3b^5+a^2b^6$.

146. The G. C. M. of three quantities is the G. C. M. of any one and the G. C. M. of the other two.

Thus whatever measure is common to the three quantities A, B, C, must plainly be common to A and the G. C. M. of B, C.

EXAMPLES.

1. Find the G. C. M. of $4xy^2z^2$, $6x^2yz^2$, and $8x^2y^2z$.

The G. C. M. of $4xy^2z^2$ and $6x^2yz^2$ is $2xyz^2$; and the G. C. M. of $2xyz^2$ and $8x^2y^2z$ is $2xyz$.

Therefore the G. C. M. required is $2xyz$.

2. Find the G. C. M. of x^4-1, x^4+2x^2-3, and $2x^4+2x^3+3x+3$.

The G. C. M. of x^4-1 and x^4+2x^2-3 is x^2-1; and the G. C. M. of x^2-1 and $2x^4+2x^3+3x+3$ is $x+1$, the G. C. M. required.

Exercises, XXXV.

Find the G. C. M. of

1. x^2+x-30, $x^2+11x+30$; and x^2-x-42.

2. $6x^2-x-2$, $21x^2-17x+2$, and $15x^2+5x-10$.

3. $12a^2+7ab-10b^2$, $15a^2+2ab-8b^2$, and $15a^2+5ab-10b^2$.

4. $10x^2-30x+20$, $15x^2-75x+90$, and $6x^2-21x+18$.

5. $x^3+3x^2y+3xy^2+y^3$, $x^3+x^2y+xy^2+y^3$, and $x^3+x^2y-xy^2-y^3$.

6. $x^4-9x^3+29x^2-39x+18$, $4x^3-27x^2+58x-39$, and $x^3-8x^2+19x-12$.

THE LEAST COMMON MULTIPLE.

147. The L. C. M. of two or more quantities is the common multiple of lowest dimensions and least numerical coefficient or coefficients.

148. To find the L. C. M. of any two quantities A and B.

I. If they contain no common measure except unity, their L. C.M. is plainly their product.

Thus the L. C. M. of $4a$ and $5b$ is $20ab$.

II. If they have a G. C. M., let it be C; so that $A=aC$ and $B=bC$, where a and b are two whole expressions having no common measure.

Then the L. C. M. of A and B is the L. C. M. of aC and bC, that is abC.

And $abC=aB=\frac{A}{C}.B$.

Therefore the L. C. M. of two quantities is equal to one of them multiplied by the quotient of the other divided by their G. C. M.

149. In the case of two mononomials the L. C. M. may be obtained by multiplying the L. C. M. of the numerical coefficients by the greatest power or powers of the several letters involved in the given quantities.

EXAMPLES.

1. The L. C. M. of $6x^2yz^2$ and $4xy^2z$ is $12x^2y^2z^2$.

2. Find the L. C. M. of x^4-1 and x^4+2x^2-3.

The G. C. M. of these quantities is x^2-1; and the quotient of x^4-1 divided by x^2-1 is x^2+1.

Therefore the L. C. M. is $(x^2+1)(x^4+2x^2-3)$.

3. Find the L. C. M. of $4x^2-4xy-8y^2$ and $6x^2-6y^2$.

The G. C. M. is $2(x+y)$, and the quotient of $6x^2-6y^2$ divided by $2(x+y)$ is $3(x-y)$.

Therefore the L. C. M. is $3(x-y)(4x^2-4xy-8y^2)$.

EXERCISES, XXXVI.

Find the L. C. M. of

1. $12a^2b^2c$ and $9ab^2c^3$. 2. $21x^6y^2z$ and $15ax^3y^4$.
3. $6(x^2y-xy^2)$ and $16(x^3-y^3)$.
4. $16x^2-20x-6$ and $2x^3-5x^2+5x-3$.
5. x^3-x^2+x-1 and x^2-1.
6. $12x^2-x-1$ and $6x^2-5x+1$
7. $3x^2-5x+2$ and $4x^3-4x^2-x+1$.

150. The L. C. M. of three quantities is the L. C. M. of any one, and the L. C. M. of the remaining two.

Thus the L. C. M. of the three quantities A, B, C, is the L. C. M. of one of them, as A, and the L. C. M. of B, C.

The L. C. M. of four quantities is the L. C. M. of any one, and the L. C. M. of the remaining three.

EXAMPLE.

Find the L. C. M. of x^2-1, x^3-1, and x^4-2x^2+1.

The L. C. M. of x^2-1 and x^3-1 is $(x^2-1)(x^2+x+1)$; and the L. C. M. of $(x^2-1)(x^2+x+1)$ and x^4-2x^2+1 is $(x^2+x+1)(x^4-2x^2+1)$.

EXERCISES, XXXVII.

Find the L. C. M. of

1. $2x^2y^2z$, $6xyz^2$, and $8x^2yz^3$.

2. x^2+x-30, $x^2+11x+30$; and x^2-x-42.

3. x^2-1, $7x^2+5x-2$, and $7x^2-5x-2$.

4. $12a^2+7ab-10b^2$, $15a^2+2ab-8b^2$, and $15a^2+5ab-10b^2$.

5. x^2-4, x^3+8, $2x^2+3x-2$, and $2x^2-3x-2$.

6. $21x^2+8xy-4y^2$, $49x^2-4y^2$, $21x^2-20xy+4y^2$, and $49x^2-28xy+4y^2$.

FRACTIONS.

151. When one quantity is not exactly divisible by another, the quotient is represented by writing them in the form of a fraction.

Thus the quotient of $+a$ divided by $-b$ is $\dfrac{+a}{-b}$, and the quotient of x^2-3x+1 divided by x^3-3 is $\dfrac{x^2-3x+1}{x^3-3}$.

152. Hence, conversely, the product of such a fraction and its denominator is equal to its numerator.

Thus $\left(\dfrac{+a}{-b}\right)(-b)=+a$, and $\left(\dfrac{x^2-3x+1}{x^3-3}\right)(x^3-3)$ $=x^2-3x+1$.

153. A fraction is not altered in value by multiplying or dividing the numerator and denominator by the same quantity. For, m being any quantity, we have, by the previous Article,

$$\left(\frac{+ma}{-mb}\right)(-mb)=+ma.$$

Also $\left(\dfrac{+a}{-b}\right)(-mb)=\left(\dfrac{+a}{-b}\right)(-b)\,m=+ma.$

That is, $\dfrac{+ma}{-mb}$ is equivalent, as a factor, to $\dfrac{+a}{-b}$;

$$\therefore \frac{+ma}{-mb}=\frac{+a}{-b}.$$

And so for other fractions.

154. A fraction which involves powers with negative indices or fractional coefficients in the terms of the numerator or denominator, can always be reduced to one whose numerator and denominator are whole expressions by multiplying the numerator and denominator by a proper quantity.

Thus, $\dfrac{x^2-\frac{1}{3}x+x^{-1}}{4x^3-\frac{1}{2}x^2-\frac{1}{3}x^{-2}}$ is reduced to the equivalent fraction $\dfrac{6x^4-2x^3+6x}{24x^5-3x^4-2}$, by multiplying numerator and denominator by $6x^2$.

In like manner, by using the multiplier $pqxy^3$,

$$\frac{\frac{a}{p}x^2y^{-2}-\frac{b}{q}x^{-1}y^3}{pxy-qx^{-1}y^{-1}} \text{ is reduced to } \frac{aqx^3-bpy^5}{p^2qx^2y^3-pq^2y}.$$

EXERCISES, XXXVIII.

Reduce to fractions whose numerators and denominators are whole expressions

1. $\dfrac{x^2}{ax^{-1}-by^{-2}}$.

2. $\dfrac{\frac{x}{a}-\frac{y}{b}}{ax+by}$.

3. $\dfrac{\frac{1}{2}x^3-2x+\frac{2}{3}}{x^4-2x^2-3x^{-1}}$.

4. $\dfrac{x-x^{-1}}{x^{\frac{2}{3}}+x^{-\frac{2}{3}}+1}$.

5. $\dfrac{ax-bx^{-\frac{1}{2}}+1}{a^{-1}x^2-bx^{-1}-1}$.

155. A fraction is said to be in *lowest terms* when its numerator and denominator contain no common measure.

A fraction, therefore, is reduced to lowest terms by dividing its numerator and denominator by their G. C. M.

EXAMPLES.

1. Reduce $\dfrac{a^2b^2}{a^2b-ab^2}$ to lowest terms.

Here ab is the G.C.M. of a^2b^2 and a^2b-ab^2; therefore the reduced fraction is $\dfrac{ab}{a-b}$.

2. Reduce $\dfrac{3x^3+x^2+x-2}{x^3-1}$ to lowest terms.

The G. C. M. of numerator and denominator is x^2+x+1; therefore the reduced fraction is $\dfrac{3x-2}{x-1}$

EXERCISES, XXXIX.

Reduce to lowest terms

1. $\dfrac{6a^3b^2}{15a^2b^3}$. 2. $\dfrac{15xy^2z}{20x^2yz^2}$. 3. $\dfrac{9bcyz}{12abxy^2}$.

4. $\dfrac{a^2by^2z-ab^2yz^2}{aby^2z^2}$. 5. $\dfrac{ac+bc}{a^2-b^2}$.

6. $\dfrac{x^2+3x+2}{x^2+x-2}$. 7. $\dfrac{3x^2+xy-10y^2}{2x^2+5xy+2y^2}$.

8. $\dfrac{2x^2+3x+1}{x^2-x-2}$. 9. $\dfrac{2x^2-2xy+x-y}{2x^2+2xy+x+y}$.

10. $\dfrac{p^2+mpq-mq^2-pq}{p^3-q^3+pq^2-p^2q}$. 11. $\dfrac{a^2-3ab+ac+2b^2-2bc}{a^2-b^2+2bc-c^2}$.

12. $\dfrac{x^4-y^4}{x^8+x^6y^2+x^2y+y^3}$. 13. $\dfrac{4x^2-xy-3y^2}{3x^3-3x^2y+xy^2-y^3}$.

156. Fractions are said to be *like* or *unlike* according as they have the same or different denominators.

Thus $\dfrac{y}{z}$, $\dfrac{x}{z}$ are like fractions, and $\dfrac{2x}{x-1}$, $\dfrac{2y}{y-1}$ unlike fractions.

Unlike fractions are reduced to like fractions by multiplying the numerator and denominator of each by the respective quotients of the L. C. M. of the denominators divided by the several denominators.

EXAMPLES.

1. Reduce $\dfrac{a}{b}$ and $\dfrac{c}{d}$ to like fractions.

Here the L. C. M. is bd, and the quotient of this divided by b is d, which is the multiplier for the first fraction. Hence $\dfrac{a}{b}=\dfrac{ad}{bd}$. In like manner $\dfrac{c}{d}=\dfrac{bc}{bd}$.

2. Reduce $\dfrac{a+b}{a-b}$ and $\dfrac{a-b}{a+b}$ to like fractions.

The L.C.M. of the denominators is a^2-b^2, the quotients of which divided by $a-b$ and $a+b$ are $a+b$, $a-b$, respectively.

$$\text{Therefore } \frac{a+b}{a-b}=\frac{(a+b)(a+b)}{(a-b)(a+b)}=\frac{a^2+2ab+b^2}{a^2-b^2};$$

$$\text{and } \frac{a-b}{a+b}=\frac{(a-b)(a-b)}{(a+b)(a-b)}=\frac{a^2-2ab+b^2}{a^2-b^2}.$$

3. Reduce $\frac{1}{x-y}$, $\frac{1}{x+y}$, and $\frac{1}{x^4-y^4}$ to like fractions.

The L.C.M. of the denominators is x^4-y^4, the quotients of which divided by $x-y$, $x+y$, and x^4-y^4 are $x^3+x^2y+xy^2+y^3$, $x^3-x^2y+xy^2-y^3$, 1, respectively; therefore the equivalent like fractions are

$$\frac{x^3+x^2y+xy^2+y^3}{x^4-y^4},\quad \frac{x^3-x^2y+xy^2-y^3}{x^4-y^4},\quad \frac{1}{x^4-y^4}.$$

EXERCISES, XL.

Reduce to like fractions

1. $\frac{1}{bc}$, $\frac{1}{ab}$.

2. $\frac{1}{yz}$, $\frac{1}{zx}$, $\frac{1}{xy}$.

3. $\frac{a}{bc}$, $\frac{b}{ca}$, $\frac{c}{ab}$.

4. $\frac{1}{x-1}$, $\frac{2}{x+1}$.

5. $\frac{1}{x^2-1}$, $\frac{1}{(x-1)^2}$.

6. $\frac{3x}{x^2+1}$, $\frac{5x^2}{x^4-1}$.

7. $\frac{2}{x}$, $\frac{3}{x-1}$, $\frac{5}{x^2-1}$.

8. $\frac{1}{(a-b)(a-c)}$, $\frac{1}{(b-c)(b-a)}$, $\frac{1}{(c-a)(c-b.)}$.

157. To find the *sum* of two or more like fractions.

Let the two fractions be $\frac{A}{C}$, $\frac{B}{C}$.

Then since $\frac{A}{C}.C=A$, and $\frac{B}{C}.C=B$, it follows that the sum of $\frac{A}{C}.C$ and $\frac{B}{C}.C=A+B$. But the sum of $\frac{A}{C}.C$ and $\frac{B}{C}.C$ is equal to the product of C, and the *sum* of $\frac{A}{C}$ and $\frac{B}{C}$, that is,

$$C\times\left(\textit{sum of } \frac{A}{C} \text{ and } \frac{B}{C}\right)=A+B.$$

Therefore the *sum* of $\frac{A}{C}$ and $\frac{B}{C}=\frac{A+B}{C}$.

In like manner may the sum of three or more like fractions be obtained.

Hence the sum of any number of like fractions is a like fraction whose numerator is the sum of their numerators.

Thus the *sum* of $\frac{x^2-3x-2}{x^2-1}$ and $\frac{2x^2+x+1}{x^2-1}$ is $\frac{3x^2-2x-1}{x^2-1}$.

So also the *difference* between two like fractions is a like fraction whose numerator is the difference between their numerators.

Thus the difference between $\frac{2x^2-xy+5y^2}{x^3-y^3}$ and $\frac{x^2+xy+3y^2}{x^3-y^3}$ is $\frac{x^2-2xy+2y^2}{x^3-y^3}$.

158. Addition and Subtraction of unlike fractions are performed by reducing the unlike to like fractions, and proceeding as in the previous Article.

159. The process may be conducted by introducing the signs + and − to denote Addition and Subtraction, these signs being combined, according to the

Distributive Law, Art. 27, with the signs of the numerators of the fractions which they precede.

Thus $+\frac{a-b}{c-d}=\frac{+a+-b}{c-d}$; $-\frac{a-b}{c-d}=\frac{-a--b}{c-d}$ $=\frac{-a+b}{c-d}$; $\frac{x-y}{2x^2-y^2}=+\frac{x-y}{2x^2-y^2}=-\frac{-x+y}{2x^2-y^2}$.

Since also a fraction is not altered by multiplying numerator and denominator by -1, it follows that

$$\frac{a-b}{c-d}=\frac{-a+b}{d-c}=-\frac{a-b}{d-c}.$$

Conversely, we have $-\frac{x-a}{b-y}=+\frac{-x+a}{b-y}=+\frac{x-a}{-b+y}$ $=+\frac{x-a}{y-b}$.

Thus also,

$$-\frac{y-z}{(a-b)(a-c)}=+\frac{-y+z}{(a-b)(a-c)}$$

$$=+\frac{y-z}{(b-a)(a}=+\frac{y-z}{a-b)(c-a)}$$

160. Sums and differences of fractions when expressed as fractions are said to be *simplified.*

EXAMPLES.

1. Simplify $\frac{2x}{x-y}+\frac{2y}{x+y}$.

The L. C. M. of the denominators is x^2-y^2.

Therefore $\frac{2x}{x-y}+\frac{2y}{x+y}=\frac{2x(x+y)}{x^2-y^2}+\frac{2y(x-y)}{x^2-y^2}$

$$=\frac{2x(x+y)+2y(x-y)}{x^2-y^2}$$

$$=\frac{2x^2+4xy-2y^2}{x^2-y^2}.$$

2. Simplify $\frac{a}{b}-\frac{a^2+2ab}{a^2+b^2}+\frac{b}{a}$.

The L. C. M. of the denominators is $ab\,(a^2+b^2)$.

Therefore

$$\frac{a}{b}-\frac{a^2+2ab}{a^2+b^2}+\frac{b}{a}=\frac{a^2\,(a^2+b^2)-ab\,(a^2+2ab)+b^2\,(a^2+b^2)}{ab\,(a^2+b^2)}$$

$$=\frac{a^4+a^2b^2-a^3b-2a^2b^2+a^2b^2+b^4}{ab\,(a^2+b^2)}$$

$$=\frac{a^4-a^3b+b^4}{ab\,(a^2+b^2)}.$$

3. Simplify $\frac{a}{(a-b)\,(a-c)}-\frac{b}{(b-c)\,(b-a)}-\frac{c}{(c-a)(c-b)}$

The L. C. M. of the denominators is $(b-c)\,(c-a)\,(a-b)$, and the multipliers for the several fractions are, therefore, $-(b-c)$, $-(c-a)$, $-(a-b)$, respectively.

Therefore $\frac{a}{(a-b)\,(a-c)}-\frac{b}{(b-c)\,(b-a)}-\frac{c}{(c-a)(c-b)}=$

$$\frac{-a\,(b-c)+b\,(c-a)+c\,(a-b)}{(b-c)\,(c-a)\,(a-b)}$$

$$=\frac{2ac-2ab}{(b-c)\,(c-a)\,(a-b)}=\frac{-2a}{(c-a)\,(a-b)}.$$

Exercises, XLI.

Simplify the expressions

1. $\frac{x^2}{3y}-\frac{y^2}{2x}$. 2. $\frac{a}{4bc}-\frac{b}{6cd}$. 3. $\frac{1}{x-2}-\frac{1}{x-3}$.

4. $\frac{2x+1}{3x+2}-\frac{4x+5}{5x+4}$. 5. $\frac{5}{2-3y}+\frac{3}{2+3y}$.

6. $\dfrac{1}{x^2-y^2}-\dfrac{1}{x^2+y^2}$. 7. $\dfrac{1}{1-a+a^2}+\dfrac{1}{1+a+a^2}$.

8. $\dfrac{1+x}{1+x+x^2}-\dfrac{1-x}{1-x+x^2}$. 9. $\dfrac{1}{x^2-xy+y^2}-\dfrac{1}{x^2+xy+y^2}$.

10. $\dfrac{x}{x+1}+\dfrac{2}{x+2}+\dfrac{3}{x+3}$. 11. $\dfrac{2}{2x+1}+\dfrac{3}{3x+2}-\dfrac{8}{4x+3}$.

12. $\dfrac{2a}{a^2-b^2}+\dfrac{1}{a+b}-\dfrac{1}{a-b}$. 13. $\dfrac{x}{1-x}-\dfrac{x^2}{(1-x)^2}-\dfrac{x^3}{(1-x)^3}$.

14. $\dfrac{3a-4b}{7}-\dfrac{2a-b-c}{3}+\dfrac{15a-4c}{12}$.

15. $\dfrac{a^2+b^2}{a^2-b^2}+\dfrac{a}{a+b}-\dfrac{b}{a-b}$.

16. $\dfrac{2}{x}+\dfrac{1+x}{1-x}-\dfrac{1-x}{1+x}$. 17. $\dfrac{a+b}{a-b}+\dfrac{a^2-b^2}{a^2+b^2}-\dfrac{a^2+b^2}{a^2-b^2}$.

18. $\dfrac{b-c}{bc}+\dfrac{c-a}{ca}+\dfrac{a-b}{ab}$.

19. $\dfrac{1}{2(x+1)}-\dfrac{4}{x+2}+\dfrac{9}{2(x+3)}$.

20. $\dfrac{3}{4(1-x)^2}+\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}-\dfrac{1-x}{4(1+x^2)}$.

21. $\dfrac{x^{2n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$.

22. $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-c)(b-a)(x-b)}$

$+\dfrac{c}{(c-a)(c-b)(x-c)}$.

161. Since a whole expression may be considered as a fraction whose denominator is unity, the sum or difference of a whole expression and a fraction may be expressed as a fraction; and, conversely, a fraction may sometimes be expressed as the sum of a whole expression and a fraction.

EXAMPLES.

1. $2x^2-\frac{1}{x}=\frac{2x^2}{1}-\frac{1}{x}=\frac{2x^3-1}{x}$.

2. $3a-1+\frac{1}{3a+1}=\frac{3a-1}{1}+\frac{1}{3a+1}=\frac{9a^2}{3a+1}$.

3. $x^2-1-\frac{x-1}{x^2+1}=\frac{x^2-1}{1}-\frac{x-1}{x^2+1}=\frac{x^4-x}{x^2+1}$.

4. $\frac{ab+c}{b}=\frac{ab}{b}+\frac{c}{b}=a+\frac{c}{b}$.

5. $\frac{2x^4-x^3+2x-3}{x^3-x+1}=2x-1+\frac{2x^2-x-2}{x^3-x+1}$.

Here the numerator $2x^2-x-2$ is the remainder when $2x^4-x^3+2x-3$ is divided by x^3-x+1.

EXERCISES, XLII.

Find the fractions equivalent to

1. $\frac{x}{y}+\frac{y}{x}-2$.

2. $ax+\frac{1}{bx}+\frac{2}{cx^2}$.

3. $x-\frac{x^2-1}{x-y}$.

4. $2x-1-\frac{3}{x+1}$.

5. $3x+2-\frac{7x+6}{2x+3}$.

6. $x^2+xy+y^2-\frac{xy-y^3}{x-y}$.

Reduce to the sum of a whole expression and a fraction

7. $\dfrac{2xy-3}{2xy}$. 8. $\dfrac{x^2-1-2y^2}{2y^2}$. 9. $\dfrac{x^3-3x^2+2}{x^2-3}$.

10. $\dfrac{6a^2-4a+5}{2a^2-a+1}$. 11. $\dfrac{a^5-a+5}{a^3+1}$.

12. $\dfrac{x^{2n+2}-3x^n+x^2+1}{x^{2n}+1}$.

162. To find the *product* of two or more fractions.

Let there be two fractions $\dfrac{A}{B}$, $\dfrac{C}{D}$.

Then since $\dfrac{A}{B}.B=A$, and $\dfrac{C}{D}.D=C$, the *product* of the four quantities $\dfrac{A}{B}$, B, $\dfrac{C}{D}$, D is equal to AC, that is $\dfrac{A}{B}.\dfrac{C}{D}.BD=AC$.

Therefore $\dfrac{A}{B}.\dfrac{C}{D}=\dfrac{AC}{BD}$.

In like manner the product of three or more fractions may be found.

Hence the product of any number of fractions is a fraction whose numerator is the product of their numerators, and denominator the product of their denominators.

163. The case of the product of a fraction and a whole expression is included in the preceding Article

by considering the whole expression as a fraction, whose denominator is unity.

EXAMPLES.

1. $\dfrac{a+b}{a} \cdot \dfrac{a^2}{a-b} = \dfrac{a^2(a+b)}{a(a-b)} = \dfrac{a^2+ab}{a-b}.$

2. $\dfrac{2a^2}{a^2-b^2} \cdot \dfrac{(a+b)^2}{4a^2b^2} = \dfrac{2a^2(a+b)^2}{4a^2b^2(a^2-b^2)} = \dfrac{a+b}{2b^2(a-b)}.$

3. $\left(3 - \dfrac{2x^2-4}{x^2-1}\right)\dfrac{x-1}{x^2+1} = \dfrac{3x^2-3-2x^2+4}{x^2-1} \cdot \dfrac{x-1}{x^2+1}$

$$= \frac{x^2+1}{x^2-1} \cdot \frac{x-1}{x^2+1} = \frac{1}{x+1}$$

EXERCISES, XLIII.

Find the product of

1. $\dfrac{x}{1-x}, \dfrac{y}{1+x}.$ 2. $\dfrac{a+bx}{a^2-x^2}, \dfrac{(a+x)^2}{ac+bcx}.$

3. $\dfrac{a^4-b^4}{a^2-2ab+b^2}, \dfrac{a-b}{a^2+ab}.$ 4. $\dfrac{2m^2n}{m^2-n^2}, \dfrac{m^2+mn}{6mn^2}.$

5. $\dfrac{ax^2}{9by^2}, \dfrac{18cxy^2}{4a^2x^3}, \dfrac{a^2-ax}{a^2-x^2}.$

6. $\dfrac{m^2-mn+n^2}{m^3-3mn(m-n)-n^3}, \dfrac{m^2-n^2}{m^3+n^3}.$

7. $\dfrac{(x-y)(x^2-xy+y^2)}{x^3+y^3}, \dfrac{(x+y)(x^2+xy+y^2)}{x^3-y^3}.$

8. $\dfrac{x^3+2x^2y+2xy^2+y^3}{x^3-y^3}, \dfrac{x-y}{x+y}.$

Simplify

9. $\dfrac{b+c}{bc}(b^2+c^2-a^2) + \dfrac{c+a}{ca}(c^2+a^2-b^2)$

$+ \dfrac{a+b}{ab}(a^2+b^2-c^2).$

10. $\left\{\dfrac{a+b}{2(a-b)} - \dfrac{a-b}{2(a+b)} + \dfrac{2b^2}{a^2-b^2}\right\}\dfrac{a-b}{2b}.$

11. $\dfrac{x^2-a^2}{x^3-a^3}\left(\dfrac{x^2}{x+a}+a\right)+\dfrac{x^2-a^2}{x^3+a^3}\left(\dfrac{x^2}{x-a}-a\right).$

12. $\left\{\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}\right\}\left\{\dfrac{x^2+y^2}{2xy}+1\right\}\dfrac{xy}{x^2+y^2}.$

13. $\left(1+\dfrac{x}{1-x}\right)\left(1-\dfrac{x}{1+x}\right)\left(1-x^2+\dfrac{1-x^2}{x}\right).$

14. $\left(1-\dfrac{b^2}{a^2}\right)\left(1-\dfrac{ab-b^2}{a^2}\right)\cdot\dfrac{a^4}{a^3+b^3}\cdot\dfrac{a-b}{a^2+b^2}.$

164. Since the *quotient* of one fraction divided by another is the fraction which, multiplied by the latter, is equal to the former, it follows that the quotient of one fraction divided by another is the product of the former, and the reciprocal of the latter.

Thus $\dfrac{A}{B}\div\dfrac{C}{D}=\dfrac{A}{B}\cdot\dfrac{D}{C}=\dfrac{AD}{BC}$, because $\dfrac{AD}{BC}\cdot\dfrac{C}{D}=\dfrac{A}{B}$.

Hence also to divide a fraction by a whole expression we must multiply its denominator by the latter.

Thus $\dfrac{A}{B}\div C=\dfrac{A}{B}\div\dfrac{C}{1}=\dfrac{A}{B}\cdot\dfrac{1}{C}=\dfrac{A}{BC}$.

EXAMPLES.

1. $\dfrac{2}{x-y}\div\dfrac{3}{x+y}=\dfrac{2}{x-y}\cdot\dfrac{x+y}{3}=\dfrac{2(x+y)}{3(x-y)}.$

2. $\dfrac{x^2-y^2}{x^4+y^4}\div\dfrac{x-y}{xy}=\dfrac{x^2-y^2}{x^4+y^4}\cdot\dfrac{xy}{x-y}=\dfrac{xy(x+y)}{x^4+y^4}.$

3. $\dfrac{(x^2-y^2)\,xy}{(a+b)^2}\div\dfrac{xy^2(x+y)}{a^2+ab}=\dfrac{(x^2-y^2)\,xy}{(a+b)^2}\cdot\dfrac{a^2+ab}{xy^2(x+y)}$

$=\dfrac{(x-y)\,a}{(a+b)y}.$

EXERCISES, XLIV.

Divide

1. $\frac{x-y}{a}$ by $\frac{x+y}{a}$.

2. $\frac{a^2b}{a^2-b^2}$ by $\frac{ab^2}{a^2+ab}$.

3. $\frac{x^3-y^3}{(x^2+y^2)^2}$ by $\frac{x-y}{x^2+y^2}$.

4. $\frac{(a-b)^3}{x^3-y^3}$ by $\frac{a^3-b^3}{x^2-xy+y^2}$.

5. $\frac{(b-c)(c-a)}{a-b}$ by $\frac{(c-a)(a-b)}{b-c}$.

6. $\frac{x^2-5x+6}{x^2-5x}$ by $\frac{x^2-3x}{x^2-6x+5}$.

7. $\frac{x^4-y^4}{x^2-2xy+y^2}$ by $\frac{x^2+xy}{x-y}$.

Simplify

8. $\left\{\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right\} \div \left\{\frac{x+y}{x-y}-\frac{x-y}{x+y}\right\}$.

9. $\left\{\frac{a^2+b^2}{a^2+ab+b^2}+\frac{a^2-b^2}{a^2-ab+b^2}\right\} \div \left\{\frac{a^2+b^2}{a^2+ab+b^2}-\frac{a^2-b^2}{a^2-ab+b^2}\right\}$.

SURDS.

165. Those roots of quantities of which the numerical values cannot be exactly expressed are called *irrational quantities* or *surds*

Thus, for example, $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{7}$ are surds.

166. When a surd is combined by Addition, Subtraction, Multiplication, or Division with a whole number or a fraction, evidently the exact value of the combination cannot be expressed. Such quantities are also called surds.

Thus, for example, $3+\sqrt{2}$, $5-\sqrt{3}$, $7\sqrt[3]{4}$, $\frac{1}{6}\sqrt[5]{10}$ are surds.

167. The quantity under the radical sign is called the *base*.

Thus 2 is the base of $\sqrt[3]{2}$, and 5 of $\sqrt[4]{5}$.

168. The number which expresses the root to be extracted is called the *radical index*.

Thus 2 is the radical index of $\sqrt{5}$, and 3 is the radical index of $5\sqrt[3]{4}$.

169. A surd whose radical index is 2 is called a *quadratic* surd, and one whose radical index is 3 a *cubic* surd.

170. *Similar* surds are those which have, or can be made to have, the same base and the same radical index.

Thus $5\sqrt[3]{2}$ and $3\sqrt[3]{2}$ are similar surds; whereas $\sqrt[3]{4}$ and $\sqrt[4]{4}$ are *dissimilar* surds.

171. Since $a=a^{\frac{n}{n}}=\sqrt[n]{a^n}$, it follows that any rational quantity can be expressed in the *form* of a surd.

Thus $2=\sqrt[3]{2^3}=\sqrt[3]{8}$ and $5=\sqrt{25}$.

172. When the radical index of a surd is the product of two or more integers, the radical sign and index may be equivalently replaced by a combination of radical signs and indices.

Thus $\sqrt[pq]{a}=a^{\frac{1}{pq}}=(a^{\frac{1}{p}})^{\frac{1}{q}}=\sqrt[q]{\sqrt[p]{a}}=\sqrt[r]{\sqrt[q]{a}}$.

In like manner $\sqrt[pqr]{a}$ may be written in the equivalent forms $\sqrt[p]{\sqrt[q]{\sqrt[r]{a}}}$, $\sqrt[q]{\sqrt[r]{\sqrt[p]{a}}}$.

Thus

$$\sqrt[8]{2}=\sqrt{\sqrt[4]{2}}=\sqrt{\sqrt{\sqrt{2}}} \quad \sqrt[6]{x}=\sqrt{\sqrt[3]{x}}=\sqrt[3]{\sqrt{x}}.$$

173. Since $a\sqrt[n]{b}=ab^{\frac{1}{n}}=(a^nb)^{\frac{1}{n}}=\sqrt[n]{a^nb}$, it follows that a rational multiplier of a surd may be brought under the radical sign by multiplying the base by the rational factor raised to the power indicated by the radical index.

Thus $5\sqrt{2}=\sqrt{5^2\times 2}=\sqrt{50}$; $2\sqrt[3]{4}=\sqrt[3]{2^3\times 4}=\sqrt[3]{32}$.

174. Conversely, a surd whose radical sign is n can be reduced to another whenever the base is a multiple of a complete nth power.

Thus $\sqrt{48}=\sqrt{4^2\times 3}=4\sqrt{3}$; $\sqrt{60}=\sqrt{2^2\times 15}=2\sqrt{15}$; $\sqrt[3]{40}=\sqrt[3]{2^3\times 5}=2\sqrt[3]{5}$.

175. Since $\sqrt[n]{a}=a^{\frac{1}{n}}=a^{\frac{m}{mn}}=\sqrt[mn]{a^m}$, it follows that the radical index may be multiplied by a factor, if the base be raised to the power indicated by that factor.

Thus $\sqrt{2}=\sqrt[6]{2^3}=\sqrt[6]{8}$; $\sqrt[4]{5}=\sqrt[12]{5^3}=\sqrt[12]{125}$; $\sqrt[3]{3}=\sqrt[12]{3^4}=\sqrt[12]{81}$.

176. Conversely, a surd whose radical index is mn can be reduced to one whose radical index is n, if the base of the former is an exact nth power.

Thus $\sqrt[4]{9}=\sqrt{3}$; $\sqrt[6]{8}=\sqrt[6]{2^3}=\sqrt{2}$; $\sqrt[8]{16}=\sqrt[8]{2^4}=\sqrt{2}$.

177. A surd is said to be in its simplest form when the base is neither a power indicated by any measure of the radical index, nor a multiple of a power indicated by the radical index.

Thus $\sqrt[3]{6}$ is in its simplest form because 6 is neither an exact cube nor a multiple of an exact cube; $\sqrt[6]{10}$

is in its simplest form because 10 is neither an exact square, cube, or sixth power, nor a multiple of an exact sixth power.

EXERCISES, XLV.

1. Bring the rational factor under the radical sign in $3\sqrt{2}, 2\sqrt[3]{7}, 3\sqrt[4]{5}$.

2. Simplify $\sqrt[3]{48}, \sqrt[3]{108}$.

3. Reduce $\sqrt{5}, \sqrt{3}$ to surds whose radical index is 6.

4. Reduce $\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{5}$ to surds whose radical index is 9.

5. Simplify $\sqrt[12]{9}, \sqrt[12]{125}$.

178. The Algebraic sum of any number of surds is expressed in its simplest form by simplifying the several terms, and collecting similar surds under one term.

179. Surds are said to be *simple* or *compound* according as they contain one or more terms.

EXAMPLES.

1. $5\sqrt{3}+\sqrt{108}=5\sqrt{3}+6\sqrt{3}=(5+6)\sqrt{3}$
$=11\sqrt{3}$

2. $\sqrt{5}+3\sqrt[3]{54}-2\sqrt[3]{16}=\sqrt{5}+9\sqrt[3]{2}-4\sqrt[3]{2}=\sqrt{5}$
$+5\sqrt[3]{2}$

EXERCISES, XLVI.

Simplify

1. $3\sqrt{2}+4\sqrt{8}-\sqrt{32}$. 2. $\sqrt{12}+\sqrt{75}-\sqrt{48}$

3. $2\sqrt[3]{4}+5\sqrt[3]{32}-\sqrt[3]{108}.$ 4. $\sqrt{108}+2\sqrt{675}.$

5. $3\sqrt[3]{54}-2\sqrt[3]{16}.$

6. $\sqrt{48}-\sqrt{3}-\frac{4}{3}\sqrt[4]{729}+2\sqrt[6]{27}$

180. To find the product of two simple surds $\sqrt[m]{a}$ and $\sqrt[n]{b}.$

Let l be the L. C. M. of m and n, so that $l=mp=nq$.

Then $\sqrt[m]{a}\ \sqrt[n]{b}=a^{\frac{1}{m}}.\,b^{\frac{1}{n}}=a^{\frac{p}{l}}.\,b^{\frac{q}{l}}=(a^p.b^q)^{\frac{1}{l}}=\sqrt[l]{a^p.b^q}$

$$=\sqrt{a^{\frac{l}{m}}.b^{\frac{l}{n}}}.$$

181. The products of simple surds are also sometimes expressed as follows:

$$\sqrt[p]{a}\sqrt[pq]{b}=a^{\frac{1}{p}}b^{\frac{1}{pq}}=(ab^{\frac{1}{q}})^{\frac{1}{p}}=\sqrt[p]{a\sqrt[q]{b}}.$$

In like manner $\sqrt[p]{a}\ \sqrt[pq]{b}\ \sqrt[pqr]{c}$ may be written $\sqrt[p]{a\ \sqrt{b\ \sqrt[r]{c}}}.$

Thus, $\sqrt{2}.\sqrt[4]{3}\ \sqrt[8]{5}=\sqrt{2\ \sqrt{3\ \sqrt{5}}}.$

Conversely, $\sqrt[p]{a\ \sqrt[q]{b\ \sqrt[r]{c}}}=\sqrt[p]{a}\ \sqrt[pq]{b}\ \sqrt[pqr]{c}$ $=\sqrt[pqr]{a^{qr}.\,b^r.\,c}.$

Thus $\sqrt{2\sqrt[3]{3\sqrt[4]{5}}}=\sqrt{2}.\sqrt[6]{3}.\sqrt[24]{5}=\sqrt[24]{2^{12}.3^4.5}.$

EXAMPLES.

1. $\sqrt[n]{a}\ \sqrt[n]{b}=\sqrt[n]{ab}.$

Here the L. C. M. of n and n is n.

2. $\sqrt{a}\,\sqrt[3]{b}=\sqrt[6]{a^3b^2}$.

3. $\sqrt[3]{18}\times 5\sqrt[3]{4}=5\sqrt[3]{72}$.

4. $5\sqrt{2}\times 6\sqrt[3]{2}=30\sqrt[6]{2^3\times 2^2}=30\sqrt[6]{32}$.

Hence the product of a simple and a compound surd, or of two compound surds, can be expressed as a rational quantity or a surd.

Examples.

1. $(2+\sqrt{3})\sqrt{3}=2\sqrt{3}+3$.

2. $(\sqrt{2}+1)(\sqrt{2}-1)=2-1=1$.

3. $(2+\sqrt{3})(3-\sqrt{5})=6-2\sqrt{5}+3\sqrt{3}-\sqrt{15}$.

4. $(\sqrt{2}-\sqrt[3]{3})(3-\sqrt[4]{3})=3\sqrt{2}-3\sqrt[3]{3}-\sqrt[4]{12}+\sqrt[12]{3^7}$.

Exercises, XLVII.

Multiply

1. $3\sqrt{5}$ by $2\sqrt{3}$. 2. $2\sqrt[3]{7}$ by $3\sqrt[3]{4}$.

3. $\sqrt{5}$ by $\sqrt[3]{6}$. 4. $5\sqrt[3]{2}$ by $2\sqrt[4]{3}$.

5. $\sqrt[4]{2}$ by $\sqrt[6]{3}$. 6. $\sqrt{5}-\sqrt{6}$ by $\sqrt{3}$.

7. $\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}$ by $\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{3}}$.

8. $1+\sqrt{3}-\sqrt{2}$ by $\sqrt{6}-\sqrt{2}$.

9. $\sqrt{2}+\sqrt[3]{3}$ by $\sqrt{2}-\sqrt[4]{3}$

10. Express as simple surds

$$\sqrt{3\sqrt{2\sqrt{3\sqrt{2}}}},\quad \sqrt{2\sqrt{3\sqrt{2\sqrt{3}}}},$$

$$\sqrt{2\sqrt{2\sqrt{2}}},\qquad \sqrt[3]{2\sqrt{2\sqrt[3]{2}}}.$$

182. the quotient of one simple surd $\sqrt[m]{a}$ divided by another $\sqrt[n]{b}$ can sometimes be expressed in the form of a simple surd.

Let l be the L. C. M. of m and n, so that $l=mp=nq$.

Then $$\frac{\sqrt[m]{a}}{\sqrt[n]{b}}=\frac{a^{\frac{1}{m}}}{b^{\frac{1}{n}}}=\frac{a^{\frac{p}{l}}}{b^{\frac{q}{l}}}=\left(\frac{a^p}{b^q}\right)^{\frac{1}{l}}=\sqrt[l]{\frac{a^p}{b^q}}.$$

Therefore the quotient will be a simple surd if a^p is a multiple of b^q.

EXAMPLES.

1. $\dfrac{\sqrt[m]{a}}{\sqrt[m]{b}}=\sqrt[m]{\dfrac{a}{b}}.$

2. $\dfrac{\sqrt[m]{a}}{\sqrt[n]{a}}=\sqrt[l]{\dfrac{a^p}{a^q}}=\sqrt[l]{a^{p-q}}$, if the L. C. M. of m and n is $l=mp=nq$.

3. $\dfrac{\sqrt{5}}{\sqrt[3]{5}}=\sqrt[6]{\dfrac{5^3}{5^2}}=\sqrt[6]{5}.$

4. $\dfrac{\sqrt[3]{4}}{\sqrt[5]{4}}=\sqrt[15]{4^2}=\sqrt[15]{16}.$

5. $\dfrac{\sqrt{6}}{\sqrt{3}}=\sqrt{\dfrac{6}{3}}=\sqrt{2}.$

6. $\frac{\sqrt[3]{18}}{\sqrt[3]{6}} = \sqrt[3]{\frac{18}{6}} = \sqrt[3]{3}$.

7. $\frac{\sqrt{2}}{\sqrt[5]{4}} = \sqrt[10]{\frac{2^5}{4^2}} = \sqrt[10]{2}$.

Exercises, XLVIII.

Divide

1. $\sqrt[3]{3}$ by $\sqrt[5]{3}$.
2. $2\sqrt{5}$ by $4\sqrt[3]{5}$.
3. $\sqrt{18}$ by $\sqrt{3}$.
4. $\sqrt[3]{24}$ by $\sqrt[3]{6}$.
5. $\sqrt{3}$ by $\sqrt[5]{9}$.
6. $\sqrt[3]{4}$ by $\sqrt[5]{2}$.
7. Shew that $\sqrt{2\sqrt[3]{2\sqrt{2}}} \div \sqrt[3]{2\sqrt{2\sqrt[3]{2}}} = \sqrt[36]{128}$.
8. Shew that $\frac{\sqrt{5\sqrt[3]{2}}\,\sqrt[3]{2\sqrt[3]{5}}}{\sqrt{2\sqrt[3]{5}}\,\sqrt[3]{5\sqrt[3]{2}}} = \sqrt[9]{2 \cdot 5}$.

183. The approximate numerical values of fractions whose denominators are surds can generally be most easily calculated by multiplying the numerator and denominator by a surd, which will render the denominator rational. The following are a few cases in which the method of rationalizing the denominators of such fractions is applied.

184. The denominator of $\frac{1}{\sqrt[n]{a}}$ may be rationalized by multiplying it by $\sqrt[n]{a^{n-1}}$, and the equivalent fraction will be $\frac{\sqrt[n]{a^{n-1}}}{a}$.

Thus, $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$; $\frac{4}{\sqrt[3]{4}} = \frac{4\sqrt[3]{4^2}}{4} = \sqrt[3]{16}$.

185. The denominators of $\frac{1}{a+\sqrt{b}}$, $\frac{1}{\sqrt{a}+\sqrt{b}}$ may be rationalized by multiplying them by $a-\sqrt{b}$, $\sqrt{a}-\sqrt{b}$ respectively.

Thus, $\frac{5}{2+\sqrt{3}}=\frac{5(2-\sqrt{3})}{4-3}=10-5\sqrt{3}$;

$$\frac{23}{5-\sqrt{2}}=\frac{23(5+\sqrt{2})}{25-2}=5+\sqrt{2};$$

$$\frac{3}{\sqrt{3}+\sqrt{2}}=\frac{3(\sqrt{3}-\sqrt{2})}{3-2}=3\sqrt{3}-3\sqrt{2}.$$

186. The fraction $\frac{1}{a+\sqrt{b}+\sqrt{c}}$ is equivalent to $\frac{a+\sqrt{b}-\sqrt{c}}{a^2+b-c+2a\sqrt{b}}$, whose denominator may be rationalized by the factor $a^2+b-c-2a\sqrt{b}$.

187. In like manner the denominator of $\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$ may be rationalized by the multipliers $\sqrt{a}+\sqrt{b}-\sqrt{c}$ and $a+b-c-2\sqrt{ab}$.

Thus, $\frac{1}{1+\sqrt{2}+\sqrt{3}}=\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}=$

$$\frac{(1+\sqrt{2}-\sqrt{3})\sqrt{2}}{4}=\tfrac{1}{4}(\sqrt{2}+2-\sqrt{6}).$$

Exercises, XLIX.

Rationalize the denominators of the following fractions:

1. $\frac{5}{\sqrt{7}}$, $\frac{2}{\sqrt[3]{5}}$, $\frac{1}{\sqrt[5]{6}}$.

2. $\frac{2}{\sqrt{7}-\sqrt{3}}$.

3. $\dfrac{1+2\sqrt{3}}{3-\sqrt{2}}$. 4. $\dfrac{2}{3+\sqrt{5}}$. 5. $\dfrac{3-\sqrt{7}}{3+\sqrt{7}}$.

6. $\dfrac{4+\sqrt{2}}{4+\sqrt{3}}$. 7. $\dfrac{\sqrt{8}+\sqrt{12}}{\sqrt{3}-\sqrt{2}}$. 8. $\dfrac{1}{2+\sqrt{3}+\sqrt{5}}$.

9. $\dfrac{1}{3-\sqrt{2}+\sqrt{3}}$. 10. $\dfrac{2}{1-\sqrt{5}-\sqrt{3}}$.

11. $\dfrac{5}{\sqrt{2}+\sqrt{5}+\sqrt{7}}$. 12. $\dfrac{2}{\sqrt{2}-\sqrt{5}-\sqrt{7}}$.

EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

188. An *equation* is the statement of the equality of two different quantities, and these quantities are called the equation's *members* or *sides*.

Thus $2x+3=7$ is an equation whose sides are $2x+3$ and 7, and $x^2+px+q=0$ is an equation whose sides are x^2+px+q and 0.

189. An *identity* is the statement of the equality of two like or different forms of the same quantity.

Thus $2a+b=2a+b, 2x+3x=5x, x^2+2ax+a^2=(x+a)^2$ are identities.

190. In the case of an identity, therefore, the equality holds for all values of the quantities involved, whereas in an equation the equality does not exist except for particular values of the quantities involved.

Thus the statement $x^2+2x+1=(x+1)^2$ holds, no matter what x is; but $5x-3=7$ holds only when $x=2$, and $x^2+6=5x$ only when $x=2$, or $x=3$.

191. The unknown quantity is generally denoted by x.

Thus in the equation $ax+b=0$, a and b are known quantities, x the unknown; and in $x^2+px+q=0$, p and q are known, x unknown.

192. Quantities which, on being substituted for the unknown, reduce the equation to an identity are said to *satisfy* the equation, and are called its *roots*.

Thus 5 is a root of $2x-3=7$, because 5 when substituted for x reduces the equation to the identity $10-3=7$. So 2 and 3 are the roots of $x^2+6=5x$, because when either is substituted for x the equation is satisfied.

193. The determination of the roots is called the *solution* of the equation.

194. An equation is said to be reduced to its simplest form when its members consist of a series of mononomials involving positive integral powers only of the unknown.

Thus $x+1=x^{\frac{1}{2}}$ is reduced, by squaring its sides, to its simplest form $x^2+2x+1=x$; and $x-x^{-1}=2$ is reduced by multiplying its sides by x to its simplest form $x^2-1=2x$.

195. Equations when reduced to their simplest forms are classified according to their order or degree.

196. The right-hand members of the standard forms are generally made 0, and the term independent of x is called the *absolute* term.

Thus -3 is the absolute term of $x^2+4x-3=0$.

197. *Simple Equations*, or those of the first degree, are those in which the highest power of the unknown quantity is the first. Their general form is

$$ax+b=0.$$

198. *Quadratic Equations*, or those of the second degree, are those in which the highest power of the unknown quantity is the second. Their general form is

$$ax^2+bx+c=0.$$

199. Equations of the third and fourth degrees are called *Cubic* and *Biquadratic Equations*, respectively, their general forms being

$$ax^3+bx^2+cx+d=0,$$
$$ax^4+bx^3+cx^2+dx+e=0.$$

And, generally, if the highest power of the unknown is the nth, the equation is said to be of the nth degree.

200. The coefficient of the highest power of x can always be made unity by dividing both sides of the equation by the coefficient of that power; so that the general forms of simple, quadratic, cubic, &c., equations may be written

$$x+p=0,$$
$$x^2+px+q=0,$$
$$x^3+px^2+qx+r=0, \text{ \&c.}$$

201. It is proved in works on the *Theory of Equations* that the number of the roots of an equation is equal to its degree; so that a simple equation has one root, a quadratic two roots, a cubic three, and so on.

202. In order to solve an equation it is generally necessary to reduce it by one or more of the following processes:

Transposition of Terms,
Clearing of Fractions,
Clearing of Surds.

These operations will be illustrated by applying them in order to the solution of simple equations.

TRANSPOSITION OF TERMS.

203. If an equation contains neither fractions nor surds, it may be solved by transposition of terms, which consists in taking the unknown quantities to one side of the equation and the known to the other side, the sign of the quantity which is so transposed being changed.

Thus if the equation is

$$ax+b=cx+d,$$

by adding $-b$ to each side we get

$$ax+b-b=cx+d-b,$$

that is $$ax=cx+d-b.$$

and so any quantity may be transposed from one side to the other by changing its sign.

Therefore $$ax-cx=d-b,$$

that is $$(a-c)\,x=d-b\,;$$

and the equation is solved by dividing each side by $a-c$, the coefficient of x.

Therefore $$x=\frac{d-b}{a-c}.$$

Thus $\frac{d-b}{a-c}$ is the root, and the equation is solved.

Examples.

1. Solve the equation $5x+10=15$.

Here $+10$ is to be taken to the right-hand side of the equation, which is then reduced to the form

$$5x=15-10=5;$$

divide by 5, the coefficient of x, and we get

$$x=1.$$

2. Solve $7x+2=11+4x$.

Here $+2$ is to be taken to the right-hand side and $+4x$ to the left-hand side; hence we get, at one step,

$$7x-4x=11-2;$$

that is $$3x=9;$$

therefore $$x=3$$

3. Solve $2(3x-1)-5(2x+1)=20-7x$.

Here the left-hand side must be reduced to a series of mononomials by removing the brackets; thus

$$6x-2-10x-5=20-7x;$$

transpose -2, -5 to the right-hand side, and $-7x$ to the left; therefore $$6x-10x+7x=20+2+5;$$

that is $$3x=27;$$

therefore $$x=9.$$

4. Solve $x(x-3)+2(1-x)=x^2-3x-5$.

Remove the brackets; thus

$$x^2-3x+2-2x=x^2-3x-5:$$

strike out x^2 and $-3x$, which are common to both sides; thus $$2-2x=-5;$$

transpose $$-2x=-5-2;$$

that is $$-2x=-7;$$

divide by -2; therefore $$x=3\tfrac{1}{2}.$$

5. Solve $(x-1)(x+1)-5x^2+3x=(3+2x)(5-2x)-20$.

Perform the multiplications indicated; thus

$$x^2-1-5x^2+3x=15+4x-4x^2-20.$$

Transpose, $x^2-5x^2+3x-4x+4x^2=15-20+1$;

that is, $-x=-4$;

therefore $x=4$.

6. Solve $a(x-a)+b(x-b)+x^2=(x-a)(x-b)$.

Perform the multiplications indicated; thu

$$ax-a^2+bx-b^2+x^2=x^2-ax-bx+ab;$$

strike out x^2, which is common to both sides, and transpose; thus $ax+bx+ax+bx=a^2+b^2+ab$;

collect coefficients of x,

$$2(a+b)x=a^2+b^2+ab;$$

divide by $2(a+b)$; therefore $x=\dfrac{a^2+b^2+ab}{2(a+b)}$.

EXERCISES, L.

1. $3x-7=x+3$.

2. $8x+9=5x+18$.

3. $16x-10=41-x$.

4. $5-19x=2+11x$.

5. $21x+7=4(x-3)+3x+61$.

6. $8(2-x)+7(1-2x)=37-28x$.

7. $3(x-5)-5(x-3)=21x-46$.

8. $2(3x-1)-17=5(x+2)-7(3x+1)$.

9. $(x-1)(x-2)+(x-1)(x-3)=2(x-2)(x-3)$.

10. $2x(2x+7)-4(x+3)-11=(1-x)(3-4x)+2x+19$.

11. $(x-1)(x-2)+(x-2)(x-3)=2(x-1)(x-3)-x$.

12. $(x+2)^2+7(3x-1)=6(2x+4)+(x-2)^2+3$.

13. $(1-2x)(1-3x)-23=(6x+1)(x-1)-3x$.

14. $ax=bx+c$.

15. $ax+b=mx+n$.

16. $x(x-a)+x(x-b)=2(x-a)(x-b)$.

17. $(x-a)(x-b)=(x-a-b)^2$.

18. $(a-x)(b-x)=(p+x)(q+x)$.

19. $(x+2a)(x-a)^2=(x+2b)(x-b)^2$.

20. $(x-a)^3(x+a-2b)=(x-b)^3(x-2a+b)$.

CLEARING OF FRACTIONS.

204. If an equation contains fractions, it may be reduced to a form capable of solution by transposition, by multiplying each side of the equation by the L. C. M. of all the denominators of the fractions.

EXAMPLES.

1. Solve $\frac{x}{2}-\frac{x}{3}=\frac{x}{5}+1$.

Here 30 is the L. C. M.

Multiply each side by 30; thus

$$15x-10x=6x+30;$$

therefore $$-x=30;$$

therefore $$x=-30$$

2. Solve $\frac{x-1}{2}+\frac{2x+3}{3}=\frac{6x+19}{8}$.

Multiply each side by 24, the L. C. M. of 2, 3, 8;

thus $$12(x-1)+8(2x+3)=3(6x+19);$$

the solution of which is $x=4\frac{1}{2}$.

3. Solve $\frac{5}{x}-\frac{3}{2x}+\frac{1}{3x}=\frac{7}{6x}+8$.

Multiply each side by $6x$, the L. C. M. of x, $2x$, $3x$, $6x$; thus

$$30-9+2=7+48x;$$

whence $$x=\frac{1}{3}.$$

4. Solve $\frac{3}{x-1}+\frac{2}{x+4}=\frac{7}{2x-2}+\frac{4}{3x+12}$.

Multiply by $6(x-1)(x+4)$, the L. C. M. of the denominators; thus

$$18(x+4)+12(x-1)=21(x+4)+8(x-1);$$

whence $$x=16.$$

5. Solve $\frac{1}{x}+\frac{b}{x+a}=\frac{1+b}{x+b}$.

Multiply by $x\,(x+a)\,(x+b)$; thus

$$(x+a)\,(x+b)+\,bx\,(x+b)=x\,(1+b)\,(x+a)\,;$$

whence, on reducing, we obtain

$$(b^2+b-ab)x=-ab\,;$$

therefore $x=\frac{-ab}{b^2+b-ab}=\frac{-a}{b+1-a}=\frac{a}{a-b-1}$.

Exercises, LI.

1. $\frac{2x}{5}+\frac{3x}{7}=\frac{x}{3}+52$.

2. $\frac{x}{3}+\frac{x}{6}=3-\frac{2x}{5}$.

3. $\frac{5x}{9}-8=74-\frac{7x}{12}$.

4. $\frac{x}{2}+\frac{x+1}{7}=x-2$.

5. $\frac{x-1}{2}-\frac{x-2}{3}=\frac{x-3}{4}$.

6. $\frac{3x+1}{13}-\frac{4x-1}{5}=\frac{2-x}{2}-\frac{2x-5}{3}$.

7. $\frac{2}{3}\,(x-8)+\frac{3}{4}\,(x-9)-\frac{5}{6}\,(x-11)\;=7-\frac{3}{8}(x-17)$.

8. $\frac{3x-7}{4x+2}=\frac{3x-14}{4x-13}$.

9. $\frac{12}{x}+\frac{1}{12x}=\frac{29}{24}.$

10. $x+1-\frac{x^2+3}{x+2}=2.$

11. $\frac{7x+16}{21}-\frac{x+8}{4x+10}=\frac{23}{70}+\frac{x}{3}.$

12. $\frac{bx}{a}-\frac{d}{c}=\frac{a}{b}-\frac{cx}{d}.$

13. $\left(\frac{a}{b}+\frac{b}{a}\right)x-\left(\frac{a}{b}-{}^{b}\right)+2x=0.$

14. $\frac{2x+3}{2x+1}+\frac{1}{3x^2}=\frac{1}{x}+1.$

15. $\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3}.$

16. $\frac{x-a}{b}+\frac{x-b}{a}=\frac{a^2+b^2}{ab}.$

17. $\frac{3x-1}{2x-1}-\frac{4x-2}{3x-2}=\frac{1}{6}.$

18. $\frac{2}{2x-3}+\frac{1}{x-2}=\frac{6}{3x+2}.$

19. $\frac{3+x}{3-x}-\frac{2+x}{2-x}-\frac{1+x}{1-x}=1.$

20. $\frac{x^2-a^2}{bx}-\frac{a-x}{b}=\frac{2x}{b}-\frac{a}{x}.$

21. $\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{x^2-ab}.$

22. $\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2ax}{a^2-b^2}.$

23. $\left(\frac{x+1}{2x+1}\right)^2=\frac{x+a}{4x+a}.$

24. $\left(\frac{x-a}{x-b}\right)^3=\frac{x-2a}{x-2b}.$

CLEARING OF SURDS.

205. If an equation contains one surd, the radical sign may be removed by bringing the surd to one side, and the remaining terms to the other, and then raising both sides to the power indicated by the radical index. The equation, if of the first degree, can then be solved by the previous methods.

EXAMPLES.

1. Solve $\sqrt{x^2-5x+3}-x=2.$

Transpose, $\sqrt{x^2-5x+3}=x+2;$

square both sides, $x^2-5x+3=x^2+4x+4;$

whence $x=-\frac{1}{9}.$

2. Solve $\sqrt[3]{x^3+3x^2-5}-x=1.$

Transpose, $\sqrt[3]{x^3+3x^2-5}=x+1;$

cube both sides,

$$x^3+3x^2-5=x^3+3x^2+3x+1;$$

whence $x=-2.$

206. If the equation contains two quadratic or two cubic surds it may be reduced to an equation contain-

ing but one surd (i.) by bringing one surd to one side, and all the other terms to the other side of the equation, and then raising both sides to the power indicated by the radical index; or (ii.) by bringing the two surds to one side, and all the other terms to the other side of the equation, and then raising both sides to the power indicated by the radical index.

EXAMPLES.

1. Solve $\sqrt{2x+3}-\sqrt{2x-3}=2.$

Employing the first method, we transpose $-\sqrt{2x-3}$ to the right-hand side; thus

$$\sqrt{2x+3}=\sqrt{2x-3}+2;$$

square both sides,

$$2x+3=2x-3+4+4\sqrt{2x-3};$$

that is, $$2=4\sqrt{2x-3};$$

whence $$x=1\tfrac{5}{8}.$$

2. Solve $\sqrt{2x-1}+\sqrt{2x+4}=5.$

Employing the second method, we square both sides as they are; thus

$$2x-1+2x+4+2\sqrt{4x^2+6x-4}=25;$$

transpose and divide by 2,

$$\sqrt{4x^2+6x-4}=11-2x;$$

whence $$x=2\tfrac{1}{2}.$$

3. Clear of surds, $\sqrt[3]{1-x}+\sqrt[3]{8+x}=3.$

Cube both sides by the formula of Art. 110; thus

$$1-x+8+x+3\sqrt[3]{(1-x)(8+x)}\{\sqrt[3]{1-x}+\sqrt[3]{8+x}\}=27;$$

substitute for $\sqrt[3]{1-x}+\sqrt[3]{8+x}$ its value 3; thus

$$9\sqrt[3]{8-7x-x^2}=18;$$

divide by 9, $\sqrt[3]{8-7x-x^2}=2;$

cube, $8-7x-x^2=8;$

$$7x+x^2=0,$$

a quadratic equation.

207. If an equation contains three quadratic or three cubic surds, it may be reduced to an equation containing but one surd by bringing two surds to one side, and the remaining surd to the other, and then raising both sides to the power indicated by the radical index.

EXAMPLES.

1. Clear of surds $\sqrt{x+4}+\sqrt{2x+9}=\sqrt{3x+25}.$

Square both sides,

$$x+4+2x+9+2\sqrt{2x^2+17x+36}=3x+25;$$

transpose and divide by 2,

$$\sqrt{2x^2+17x+36}=6;$$

square, $2x^2+17x+36=36;$

that is, $2x^2+17x=0,$

a quadratic equation.

2. Clear of surds $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=0.$

Transpose, $\sqrt[3]{a}+\sqrt[3]{b}=-\sqrt[3]{c};$

cube both sides,

$$a+b+3\sqrt[3]{ab}\{\sqrt[3]{a}+\sqrt[3]{b}\}=-c;$$

substitute for $\sqrt[3]{a}+\sqrt[3]{b}$ its value $-\sqrt[3]{c}$,

$$a+b-3\sqrt[3]{abc}=-c;$$

transpose, $a+b+c=3\sqrt[3]{abc};$

cube both sides, $(a+b+c)^3=27abc.$

EXERCISES, LII.

Solve or simplify

1. $\sqrt{4x^2+3x-16}=2x+2.$

2. $\sqrt{x^2+2x+3}=x-5.$

3. $\sqrt{x^2+3x+4}+10=x.$

4. $\sqrt{(x-2)\,(x-3)}+3=2\,(x-3)-(x-2).$

5. $\sqrt{x+4}+\sqrt{x+15}=11.$

6. $\sqrt{x-5}-\sqrt{x-4}=1.$

7. $\sqrt{x-5}+\sqrt{x+5}=\sqrt{4x-6}.$

8. $\sqrt{x+3}+\sqrt{x+8}=2\sqrt{x}.$

9. $\sqrt{x+5}+\sqrt{x-3}=2\sqrt{x}.$

10. $\sqrt{x^2+ax+b^2}+c=x.$

11. $\sqrt{ax}-\sqrt{bx}=a-b.$

12. $\sqrt{x}+\sqrt{x+1}=\dfrac{2}{\sqrt{x+1}}.$

13. $\sqrt{x}+\sqrt{x+3}\quad\dfrac{5}{\sqrt{x+3}}.$

14. $\sqrt{x}-\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$

15. $\dfrac{\sqrt{12x+1}+\sqrt{12x}}{\sqrt{12x+1}-\sqrt{12x}}=18.$

16. $\dfrac{5x-1}{\sqrt{5x}+1}=1+\dfrac{\sqrt{5x}-1}{2}$

17. $\sqrt{\frac{x}{4}+3}-\sqrt{\frac{x}{4}-3}=\sqrt{\frac{2x}{3}}$.

18. $x^2+a\sqrt{x^2-b^3}=x\{a+\sqrt{x^2-b^2}\}$.

PROBLEMS.

208. When a question is assigned for solution the unknown quantity or number is generally involved in the various conditions which are proposed for its determination. The expression of these conditions in Algebraical language leads to an equation, the solution of which will be the solution of the question.

209. In some cases, although there are more unknowns than one, they are related to each other in such a manner that when one is determined the others become immediately known. In such cases the unknowns can be expressed in terms of one unknown.

Thus, if the sum of two unknowns is equal to 8 we may denote one of them by x, and the other by $8-x$; if the greater of two unknowns exceeds the less by 3, the former may be denoted by x, and the latter by $x-3$; if the product of two unknowns is equal to 12, one of them may be denoted by x, and the other by $\frac{12}{x}$; if there be two numbers, of which one exceeds 4 times the other by 7, the former may be denoted by $4x+7$, and the latter by x.

In like manner; if there are three unknowns, of which the first exceeds the second by 3, and the second exceeds the third by 5, the first may be denoted by x, the second by $x-3$, and the third by $x-8$; if there are three unknowns, which are to each other as the numbers 1, 3, 5, they may be denoted by x, $3x$, $5x$.

210. The following examples will illustrate the method of solving problems by means of simple equations of one unknown.

1. What is the height of a house wall in which a window 6 feet high has under it $\frac{1}{3}$, and above it $\frac{1}{2}$ of the whole height?

Let the height sought $=x$ feet.

Then under the window there are $\frac{1}{3}x$ feet, and above it $\frac{1}{2}x$ feet;

$$\therefore \tfrac{1}{3}x+6+\tfrac{1}{2}x=x;$$

$$x=36.$$

2. How may a debt of £5 be paid with 29 coins, some of them crowns, and the rest florins?

Let there be x crowns; then there are $29-x$ florins.

The value of the x crowns is $5x$ shillings, and the value of the $29-x$ florins is $2(29-x)$ shillings;

$$\therefore 5x+2(29-x)=100;$$

$$x=14,$$

$$29-x=15.$$

Thus a debt of £5 can be paid in the required way only with 14 crowns and 15 florins.

3. If A can perform a given work in 120 days, and B in 80 days, in how many days will A and B, working together, be able to perform it?

Let the whole work done be denoted by w, and the required number of days by x; then

amount of work done by A in one day $= \frac{w}{120}$;

" work done by B in one day $= \frac{w}{80}$;

" work done by A and B in one day $= \frac{w}{x}$;

$$\text{therefore } \frac{w}{x} = \frac{w}{120} + \frac{w}{80};$$

$$\text{divide by } w, \frac{1}{x} = \frac{1}{120} + \frac{1}{80};$$

$$x = 48.$$

Thus A and B, working together, can do the work in 48 days.

4. A number consists of two digits, the first of which is greater than the second by unity, and the sum of the digits is one-sixth of the number itself.

Let x denote the second digit, then $x+1$ will denote the first.

The number is, therefore, $10(x+1)+x = 11x+10$; and the sum of the digits is $2x+1$;

$$\text{therefore } \frac{11x+10}{6} = 2x+1;$$

$$x = 4,$$

$$x+1 = 5.$$

Thus the number is 54.

5. One hundredweight (112 lbs.) of bronze contains by weight 70 per cent. of copper, and 30 per cent. of tin; with how much copper must it be melted in order to contain 84 per cent. of copper?

Let the amount in lbs. of copper be denoted by x; then in the cwt. there will be $\frac{70}{100} \cdot 112$, or 78·4 lbs. of copper, and $\frac{30}{100} \cdot 112$, or 33·6 lbs. of tin.

Therefore the whole amount of copper in mixture will be $78{\cdot}4 + x$, and this is to be 84 per cent. of the mixture, which weighs $112 + x$ lbs.;

$$\text{therefore } 78{\cdot}4 + x = \frac{84}{100}(112 + x);$$

$$x = 98.$$

Thus 98 lbs. of copper must be added to the cwt.

6. To find at what time between h and $h+1$ o'clock the minute-hand is m minute divisions before the hour-hand.

Let x denote the number of minute divisions between the mark h and the position of the hour-hand; then the number of minute divisions between the mark h and the minute-hand will be $m+x$, and between the mark 12 and the minute-hand $5h+m+x$.

Therefore the number of minute divisions passed over by the minute-hand since h o'clock is $5h+m+x$, and the number of minute divisions passed over in the same time by the hour-hand is x; but, in the same time, the minute-hand passes over 12 minute divisions for the hour-hand's one;

$$\text{therefore } 5h+m+x=12x;$$

$$x=\frac{5h+m}{11};$$

therefore the required time is $5h+\dfrac{5h+m}{11}+m$, or $\dfrac{12}{11}(5h+m)$ minutes past h.

Thus the time between 2 and 3 o'clock when m is 15, that is, when the hour and minute-hands are at right-angles to each other, is $\dfrac{12}{11}(10+15)$, or $27\frac{3}{11}$ minutes past 2.

Exercises, LIII.

1. The sum of two numbers is 10, and their difference 4; find the greater number.

2. Divide 30 into two parts, such that one may be two-thirds of the other.

3. The difference of two numbers is 3, and their product exceeds the square of the less by 12; find them.

4. Find the number to which, if its third part be added, the sum will exceed its half by 5.

5. The denominator of a certain fraction is one less than the numerator, and twice the fraction added to three times its reciprocal makes 5; find the fraction.

6. Divide £34 4s. into two parts, such that the number of crowns in the one may equal the number of shillings in the other.

7. A and B sat down to play. A had seven shillings more than B, but, after losing ten shillings, finds that he has only half as much as B. How much money had A and B originally?

8. A person invests two-thirds of his property at 4 per cent., one-fourth at 3 per cent., and the remainder at 2 per cent.; his income is £430; what is his property?

9. If B gave half of his money to A he would have only a quarter as much as A; but if A gave B £50 he would have only half as much as B. How much have A and B, respectively?

10. A and B set out at the same time to meet each other. A, travelling $5\frac{1}{2}$ miles an hour, meets B travelling only $3\frac{1}{2}$ miles an hour, 3 miles beyond a midway station; what is the distance of the points from which they started?

11. How much wine at 15s. a gallon must be mixed with 20 gallons of wine at £1 a gallon to make a mixture worth 17s. a gallon?

12. A mixture is made of a gallons at p shillings, b gallons at q shillings, and c gallons at r shillings; what will be the value per gallon of the mixture?

13. A cistern is supplied from two taps, by one of which it is filled in 45 minutes, and by the other in

75 minutes; in what time will it be filled by both together?

14. A starts on a journey 20 minutes before B; A walks at the rate of 4 miles an hour, and B at the rate of $4\frac{1}{2}$ miles an hour; at what distance along the road will B overtake A?

15. A, who walks at the rate of $3\frac{3}{4}$ miles per hour, starts 18 minutes before B; at what rate per hour must B walk to overtake A at the ninth mile-stone?

16. A and B start to run to a flag-staff 450 yards off, and back. A returning, meets B 30 yards from the flag-staff, and arrives at the starting-point half-a-minute before B; how long did A take to run the whole distance?

17. Divide 24 into two parts, such that their sum shall be to their difference as 3 to 2.

18. Divide 30 into two parts, such that their sum shall be to the difference of their squares as 1 to 6.

19. A number consists of two digits, the first of which is less than the second by 2, and if the difference of the squares of the digits be subtracted from the number itself the remainder is 19; find the number.

20. A bill of £100 was paid with 202 coins, consisting of crown pieces and half-guineas; how many of each were used?

21. What is the first time after 7 o'clock when the hour and minute hands of a watch are exactly opposite?

22. A watch gains as much as a clock loses, and 1798 hours by the clock are equivalent to 1802 hours by the watch; find the error in each per hour.

23. At what times will the hour and minute hands of a clock be together during 12 hours?

24. A hare pursued by a greyhound is 60 leaps in

advance, and makes 9 leaps while the hound makes 6, but 3 of the hound's are equal to 7 of the hare's. How many leaps must the hound take to catch the hare?

25. A steamboat which can travel at the rate of a miles an hour, in still water, goes from one station to another with the current in t hours, and goes back in t' hours; find the velocity of the current in miles per hour.

26. In the previous question, if the distance between the stations be $19\frac{1}{2}$ miles, the time of going down the river 1 hour 18 minutes, and up the river 2 hours 10 minutes, calculate the velocity of the current, and the rate of the steamer in still water.

27. A certain number of sovereigns, shillings, and sixpences together amounts to £8 6s. 6d., and the amount of the shillings is a guinea less than that of the sovereigns, and a guinea and a half more than that of the sixpences; find the number of each coin.

28. Two minutes after a railway train has left a station, A, where it had stopped 7 minutes, it meets the express, which set out from a station, B, when the former was 28 miles on the other side of A; the express travels at double the rate of the other, and performs the journey from B to A in an hour and a half: find the rates at which the trains travel.

29. The circumference of the fore wheel of a carriage is 10 feet, and that of the hind wheel 12 feet; the former has made 100 more revolutions than the latter: how many times has the hind wheel revolved?

30. The epitaph of Diophantus, the celebrated mathematician, states that he passed the sixth part of his life in childhood, and the twelfth part in the state of youth; that, after an interval of 5 years more than one-seventh of his life, he had a son who died when he had attained to half the age of his father, and that the father survived the son four years. Find, from these data, the age of Diophantus.

31. A number consists of two digits, of which the first exceeds the second by 4; and when the digits are reversed in order, a number is obtained which is four-sevenths of the former. Find the number.

32. A steamer makes a journey of 2,568 miles in 9 days; for 3 days she is retarded by winds and currents at the rate of 3 miles an hour; for 4 days she is helped at the rate of 2 miles an hour; and for the remainder of the time her speed is due solely to her steaming power. What is her rate in still wind and water?

33. The hour is between 2 and 3 o'clock, and the minute hand is in advance of the hour hand by $14\frac{1}{2}$ minute spaces of the dial. What o'clock is it?

34. A man at his death leaves £5,850 to be divided among his family, which consists of 3 sons, 4 daughters, and his widow. Twice the widow's share is to be equal to the share of a son and a daughter, and the share of two sons is to be equal to that of three daughters. Find each person's share.

35. A quantity of leaden shot is shaken on a sieve; twice as many grains go through as are left behind; what remains is shaken on another sieve, when three times as many go through as are left; what remains is shaken on a third sieve, when four times as many pass through as are left. The number of grains which is finally left is 100. Find the number of grains of each size.

36. In one specimen of gunpowder there is n per cent. of nitre, s of sulphur, and c of charcoal; in another n', s', and c' of these ingredients, respectively. If w lbs. of the first be mixed with w' lbs. of the second specimen, what will be the per-centages of each material in the mixture?

37. Gun-metal is composed of 90 per cent. of copper and 10 per cent. of tin. Speculum metal contains 67 per cent. of copper and 33 of tin. How many cwt. (112 lbs.) of the latter should be melted with 8 cwt. of

the former in order to make an alloy in which there is three times as much copper as tin?

38. A garrison of 500 men is provisioned for 60 days. On the 14th day they lose 80 men in a sortie; on the 34th day they lose, by the explosion of a mine, 20 men and 2,000 rations; after a week they receive a reinforcement which enables them, by reducing the rations one-third, to prolong the defence until the 71st day, when they are relieved. What was the number of men in the reinforcement?

39. One half of a population can read; of the remainder, 42 per cent. can read and write; of the remainder again, 16 per cent. can read, write, and cipher, while 243,600 can neither read, write, nor cipher. What is the population?

40. A person possessed of £5,222, invested a part of his property in 5 per cent. stock, which he bought at 105, and the rest in 3 per cent. consols, at 96. How much did he invest in each kind of stock, if his annual income amounts to £191 16s. 8d.?

QUADRATIC EQUATIONS CONTAINING ONE UNKNOWN.

211. The general form of a quadratic equation is

$$x^2+px+q=0,$$

in which p and q are supposed to be known, and x an unknown quantity, whose value is to be expressed in terms of p and q.

212. Quadratic equations are called *adfected* or *pure* according as the term involving the first power of the unknown quantity does or does not appear.

Thus $x^2-2x+3=0$, $5x^2-6x=0$, are adfected quadratics; $2x^2-5=0$, $ax^2+b=0$, are pure quadratics.

213. The solution of a quadratic equation, whose right-hand member is zero, can always be immediately effected if the left-hand member is in the form of the product of two factors, each involving the first power of the unknown. For if the product of two factors be zero, one or other factor must vanish.

Thus, if $(x-1)(x-2)=0$, it follows that either $x-1=0$, and therefore $x=1$; or $x-2=0$, and therefore $x=2$.

Thus the roots are 1 and 2.

EXAMPLES.

1. Solve $x^2-2x=0$.

Here $x(x-2)=0$;
therefore, either $x=0$; or $x-2=0$, that is $x=2$.

Thus the roots are 0 and 2.

2. Solve $(2x-3)(3x+1)=0$.

Here we have $2x-3=0$; and therefore $x=\frac{3}{2}$;
or $3x+1=0$; and therefore $x=-\frac{1}{3}$.

Thus the roots are $\frac{3}{2}$ and $-\frac{1}{3}$.

3. Solve $(ax+b)(cx+d)=0$.

Here $ax+b=0$; and therefore $x=-\dfrac{b}{a}$;
or $cx+d=0$; and therefore $x=-\dfrac{d}{c}$.

Thus the roots are $-\dfrac{b}{a}$ and $-\dfrac{d}{c}$.

214. The substitution of each root gives rise in general to two different identities.

Thus, in the last example, if $-\frac{b}{a}$ be substituted in the equation for x, the identity will be

$$\left(-a.\frac{b}{a}+b\right)\left(-c.\frac{b}{a}+d\right)=0;$$

but if $-\frac{d}{c}$ be substituted for x, the identity will be

$$\left(-a.\frac{d}{c}+b\right)\left(-c.\frac{d}{c}+d\right)=0.$$

215. A pure quadratic, as $x^2-a^2=0$, may also be immediately solved by transposing and extracting the square root.

Thus, $x^2=a^2$.

Here, on extracting the square root, we get

$$+x=+a;$$

$$+x=-a;$$

$$-x=+a;$$

$$\text{or, } -x=-a;$$

amongst which equations, it will be observed, the first and last are equivalent, as are also the second and third. Hence the four equations may be combined into the two

$$x=+a;$$

$$\text{or, } x=-a;$$

and the solution may be written in the form

$$x=\pm a.$$

216. If the given quadratic be $ax^2+b=0$, we have

$$ax^2=-b;$$

$$x^2=-\frac{b}{a}.$$

If now in $-\frac{b}{a}$, a and b have opposite signs, the right-hand side $-\frac{}{a}$ will be positive, and x will be either positive or negative. If, however, a and b have the same sign, x can neither be positive nor negative, since the square of any positive or negative quantity is positive. In such a case x is said to be *imaginary* and the two imaginary roots are written in the form

$$x = \pm\sqrt{-\frac{b}{a}}.$$

Conversely, it will follow that the square of $+\sqrt{-\frac{}{a}}$, or $-\sqrt{-\frac{b}{a}}$ will be $-\frac{b}{a}$.

Thus the square of $+\sqrt{-1}$, or $-\sqrt{-1}$ is -1.

217. The solution of an adfected quadratic, as

$$x^2+px+q=0,$$

is effected as follows:

transpose, $x^2+px=-q$;

add $\frac{p^2}{4}$, the square of one-half the coefficient of x, to both sides,

$$x^2+px+\frac{p^2}{4}=\frac{p^2-4q}{4}.$$

The left-hand side is now a complete square, and the equation may be written

$$\left(x+\frac{p}{2}\right)^2=\frac{p^2-4q}{4}.$$

Extract the square root,

$$x+\frac{p}{2}=\pm\frac{1}{2}\sqrt{p^2-4q};$$

transpose $x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}.$

Thus the roots are $-\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q}$, and $-\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q}$.

The preceding method is called the *Italian* method, having been used by the Italians, who first introduced a knowledge of Algebra into Europe.

218. If the given equation be of the form

$$ax^2+bx+c=0,$$

it may be reduced to the standard form by dividing both sides by a, thus

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

which equation may be solved by the Italian method.

219. The equation $ax^2+bx+c=0$ may also be solved by the following, called the *Hindoo* method:

transpose, $ax^2+bx=-c;$

multiply each side by $4a$,

$$4a^2x^2+4abx=-4ac;$$

add b^2 to each side,

$$4a^2x^2+4abx+b^2=b^2-4ac.$$

The left-hand side is now the square of $2ax+b$, and the equation may be written

$$(2ax+b)^2=b^2-4ac.$$

Extract the square root

$$2ax+b=\pm\sqrt{b^2-4ac};$$

transpose and divide by $2a$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

EXAMPLES.

1. Solve $x^2+9=0.$

Transpose $x^2=-9;$

extract the square root, $x=\pm\sqrt{-9}=\pm3\sqrt{-1}.$

Thus the roots $+3\sqrt{-1}$, $-3\sqrt{-1}$ are imaginary.

2. Solve $x^2+4x+3=0.$

Transpose $x^2+4x=-3;$

add to each side 4, which is the square of one-half the coefficient of x,

$$x^2+4x+4=1;$$

extract the square root,

$$x+2=\pm 1;$$

therefore, $x=-2\pm 1=-1$, or -3.

Thus the roots are -1 and -3.

3. Solve $x=1+\frac{1}{x}.$

Clear of fractions and transpose,

$$x^2-x=1;$$

add to each side $\frac{1}{4}$, the square of one-half the coefficient of x,

$$x^2-x+\tfrac{1}{4}=\tfrac{5}{4};$$

extract the square root,

$$x-\frac{1}{2}=\pm\frac{\sqrt{5}}{2};$$

therefore $x=\frac{1}{2}\pm\frac{\sqrt{5}}{2}.$

Thus the roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$

4. Solve $6x^2+5x-21=0$ by the Hindoo method.

Transpose, $6x^2+5x=21;$

multiply each side by $4\times 6=24$,

$$144x^2+120x=504;$$

add $5^2=25$ to each side,

$$144x^2+120x+25=529;$$

extract the square root,

$$12x+5=\pm 23;$$

therefore $x=\frac{-5\pm 23}{12}=\frac{3}{2}$, or $-\frac{7}{3}.$

Thus the roots are $\frac{3}{2}$ and $-\frac{7}{3}$.

5. Solve $$\frac{2x-1}{2x+1}+\frac{3x+4}{3x-4}=\frac{20}{11}.$$

Clear of fractions, $12x^2+100x=-168$;
divide by 4, $3x^2+25x=-42$;
multiply each side by $4\times 3=12$,

$$36x^2+300x=-504;$$

add $25^2=625$ to each side,

$$36x^2+300x+625=121;$$

extract the square root,

$$6x+25=\pm 11;$$

$$\text{therefore } x=\frac{-25\pm 11}{6}=-2\tfrac{1}{3}, \text{ or } -6.$$

6. Solve $$3x=\frac{41}{10-3x}.$$

Clear of fractions and transpose,

$$9x^2-30x=-41;$$

multiply each side by $4\times 9=36$, and add 900,

$$324x^2-1080x+900=-1476+900=-576;$$

extract the square root,

$$18x-30=\pm\sqrt{-576}=\pm 24\sqrt{-1};$$

therefore $$x=\frac{5\pm 4\sqrt{-1}}{3}.$$

EXERCISES, LIV.

Solve

1. $(x-1)(x-2)=0$.
2. $2x^2-7x=0$.
3. $(x+3)(x-5)=0$.
4. $(2x-5)(x+3)=0$.
5. $(6x-1)(2x+3)=0$.
6. $5x(7x-3)=0$.
7. $2x^2-8=0$.
8. $3x^2+12=0$.
9. $x^2-8x+15=0$.
10. $x^2+9x+14=0$.

11. $x^2-x-12=0.$ 12. $x^2+x-20=0.$

13. $6x^2-5x+1=0.$ 14. $30x^2-x-1=0.$

15. $15x^2+7x-4=0.$ 16. $35x^2+31x+6=0.$

17. $x+\frac{1}{x}-\frac{5}{2}=0.$ 18. $\frac{x-3}{x+4}+\frac{x+2}{x-2}=\frac{23}{10}.$

19. $\frac{1}{x^2-1}+\frac{1}{x-1}+\frac{1}{x+1}=\frac{7}{8}.$

20. $\frac{5x-2}{3x+1}=\frac{3x+10}{4x-5}.$

21. $\frac{1}{x+a+b}=\frac{1}{x}+\frac{1}{a}+\frac{1}{b}.$

22. $\frac{1}{ax+4}+\frac{ax-4}{16}=1.$

23. $\frac{1}{ax+1}+\frac{1}{ax-1}=\frac{1}{\sqrt{n^2-1}}.$

24. $\frac{x+1}{x-1}-\frac{x-1}{x+1}=2a.$

25. $\sqrt{4x+17}+\sqrt{x+1}=4.$

26. $\sqrt{5x}+\sqrt{2x+2}=\sqrt{x+2}.$

27. $acx^2-bcx+adx-bd=0.$

28. $\frac{2x-1}{3x+2}+\frac{2x-3}{2x+1}=\frac{3x-2}{6x^2-x-2}.$

29. $\frac{x^4+5x^3-2x+1}{x^2+2x-1}=x^2+3x+2.$

30. $2\sqrt{3x+7}=9-\sqrt{2x-3}.$

31. $\sqrt{6x+1}+\sqrt{x+4+\sqrt{6x+1}}=2.$

220. Equations of a higher order than the second can be solved either partially or wholly by the aid of quadratics when they can be thrown into one of the following forms:—

I. $(cx+d)(ax^2+bx+c)=0$.

II. $(ax^2+bx+c)(a'x^2+b'x+c')=0$.

III. $x^4+ax^3+bx^2+ax+1=0$.

IV. $(ax^2+bx+c)^{2n}+p(ax^2+bx+c)^n+q=0$.

221. I. The equation $(cx+d)(ax^2+bx+c)=0$ is satisfied by equating either factor of the left-hand side to zero. We thus get a simple and a quadratic equation, whose roots will be the three roots of the given cubic.

EXAMPLES.

1. Solve $x^3-5x^2+6x=0$.

This equation may be thrown into the form

$$x(x^2-5x+6)=0;$$

therefore, either $x=0$, or $x^2-5x+6=0$.

The roots of the latter equation are 2, 3.

Hence the three roots of the cubic are 0, 2, 3.

2. Solve $x^3-1=0$.

This equation may be written

$$(x-1)(x^2+x+1)=0;$$

and, therefore, $x-1=0$,

$$x=1;$$

or $x^2+x+1=0$,

$$x=\frac{-1\pm\sqrt{-3}}{2}.$$

Hence the three roots are

$$1,\ \frac{-1+\sqrt{-3}}{2},\ \frac{-1-\sqrt{-3}}{2}.$$

222. II. The biquadratic $(ax^2+bx+c)(a'x^2+b'x+c')=0$ can be fully solved by equating each

dratic factor of the left-hand side to zero. We thus get two quadratic equations, each of which has two roots.

EXAMPLES.

1. Solve $(x^2-1)(x^2+x-6)=0$.

Here we have $x^2-1=0$, and therefore $x=\pm 1$;

or $x^2+x-6=0$, and therefore $x=2$, or -3.

Hence the four roots are $1, -1, 2, -3$.

223. III. The biquadratic $x^4+ax^3+bx^2+ax+1=0$ may be solved as follows:

Divide both sides by x^2,

$$x^2+ax+b+\frac{a}{x}+\frac{1}{x^2}=0.$$

$$\text{or, } x^2+\frac{1}{x^2}+a\left(x+\frac{1}{x}\right)+b=0.$$

Add 2 to each side,

$$\left(x+\frac{1}{x}\right)^2+a\left(x+\frac{1}{x}\right)+b=2.$$

Let $$x+\frac{1}{x}=y;$$

$$\text{hence } y^2+ay+b=2;$$

from which two values can be found for y, that is, for $x+\frac{1}{x}$. If these values be called p and q we have thus two quadratics,

$$x+\frac{1}{x}=p;$$

$$x+\frac{1}{x}=q;$$

the roots of which will be the roots of the given biquadratic.

Example.

Solve $12x^4-104x^3+209x^2-104x+12=0.$

By dividing both sides by 12 the equation will be reduced to the required form; thus,

$$x^4-\frac{26}{3}x^3+\frac{209}{12}x^2-\frac{26}{3}x+1=0;$$

divide by x^2, $x^2-\dfrac{26}{3}x+\dfrac{209}{12}-\dfrac{26}{3}\cdot\dfrac{1}{x}+\dfrac{1}{x^2}=0;$

add 2 to each side,

$$\left(x+\frac{1}{x}\right)^2-\frac{26}{3}x+\left(\frac{1}{x}\right)+\frac{209}{12}=2;$$

write y for $x+\dfrac{1}{x}$,

$$y^2-\frac{26}{3}y+\frac{185}{12}=0;$$

from which we obtain $y=\dfrac{37}{6}$, or $\dfrac{5}{2}$.

Hence $x+\dfrac{1}{x}=\dfrac{37}{6}$, or $x+\dfrac{1}{x}=\dfrac{5}{2}$.

The roots of the former are 6, $\frac{1}{6}$, and of the latter 2, $\frac{1}{2}$.

Hence the four solutions of the given equation are $\frac{1}{6}$, $\frac{1}{2}$, 2, 6.

224. IV. The equation $(ax^2+bx+c)^{2n}+p(ax^2+bx+c)^n+q=0$ is reduced by writing y for $(ax^2+bx+c)^n$ to the form

$$y^2+py+q=0;$$

the solution of which gives two values of y, d and e, suppose;

thus $(ax^2+bx+c)^n=d,$

or, $(ax^2+bx+c)^n=e;$

extract the nth root of both sides,

$$ax^2+bx+c=\sqrt[n]{d},$$

or, $$ax^2+bx+c=\sqrt[n]{e};$$

two quadratics which may be solved by the usual method.

It is to be observed, however, that if n be an integer, there will be really $2n$ quadratics instead of 2. For it is proved in the *Theory of Equations* that the quantity $\sqrt[n]{d}$ has n values, and therefore there will be n quadratics corresponding. In the following examples, however, we shall generally assume $\sqrt[n]{d}$ to have but one value.

EXAMPLES.

1. Solve $$x^6+19x^3-216=0.$$

Here, writing y for x^3, we get

$$y^2+19y-216=0;$$

from which we obtain $y=-27$, or 8.

Thus $$x^3=-27, \text{ or } x^3=8;$$

whence $$x=-3, \text{ or } x=2.$$

2. Solve $$x^2+24=12\sqrt{x^2+4}.$$

This equation can be written in the form

$$(x^2+4)-12\sqrt{x^2+4}=-20;$$

add 36 to each side,

$$(x^2+4)-12\sqrt{x^2+4}+36=16;$$

extract the square root,

$$\sqrt{x^2+4}-6=\pm 4;$$

whence, $$\sqrt{x^2+4}=10, \text{ or } 2;$$

square, $$x^2+4=100, \text{ or } 4;$$

$$x^2=96, \text{ or } 0.$$

Hence the four roots are 0, 0, $4\sqrt{6}$, $-4\sqrt{6}$.

3. Solve $12\left(\frac{x+1}{x-1}\right)^2+7\left(\frac{x+1}{x-1}\right)=12.$

This equation may be solved by finding $\frac{x+1}{x-1}$ first;

thus, $$\left(\frac{x+1}{x-1}\right)^2+\frac{7}{12}\left(\frac{x+1}{x-1}\right)=1;$$

add $\left(\frac{7}{24}\right)^2$,

$$\left(\frac{x+1}{x-1}\right)^2+\frac{7}{12}\left(\frac{x+1}{x-1}\right)+\left(\frac{7}{24}\right)^2=\frac{625}{576};$$

extract the square root,

$$\frac{x+1}{x-1}+\frac{7}{24}=\pm\frac{25}{24};$$

whence $$x=-7, \text{ or } \tfrac{1}{7}.$$

EXERCISES, LV.

Solve

1. $x^4+2x^2-24=0.$

2. $x^3+x^{\frac{3}{2}}-72=0.$

3. $x^3+1=0.$

4. One root of $x^3-11x^2+38x-40=0$ is 2: find the remaining roots.

5. $(x-\frac{1}{x})^2+\frac{5}{6}(x-\frac{1}{x})=1.$

6. $(x^2+x-2)^2-13(x^2+x-2)+36=0.$

7. $x^2-2x+6\sqrt{x^2-2x+5}=11.$

8. $2x+17=9\sqrt{2x-1}.$

9. $12x^4-91x^3+194x^2-91x+12=0.$

10. $18x^4-171x^3+406x^2-171x+18=0.$

11. One root of $x^3-21x^2+143x-315=0$ is 5: find the remaining roots.

12. One root of $x^3-12x^2+47x-60=0$ is 4: find the remaining roots.

13. $\sqrt{3x^2+2x+4}=6x^2+4x-622.$

14. $\sqrt{5x^2+6x-2}=15x^2+18x-30.$

15. $\sqrt[3]{x+22}-\sqrt[3]{x+3}=1.$

16. $\dfrac{x+\sqrt{2-x^2}}{x-\sqrt{2-x^2}}=\dfrac{4}{3}.$

17. $\dfrac{x+\sqrt{9-x}}{x-\sqrt{9-x}}=\dfrac{7}{3}.$

18. $\sqrt{3x^2+8x+14}+\sqrt{3x^2-8x+14}=8x.$

19. $\sqrt{3x^2+8x+14}+\sqrt{3x^2-8x+14}=\sqrt{40x^2+24}.$

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

225. The conditions of a problem are sometimes such that their expression in Algebraical language leads to a quadratic equation. The roots of this quadratic may both satisfy the conditions of the given problem, as in the following example:

Find two numbers such that their sum shall be 15, and the sum of their squares 113.

Let x denote one number; then $15-x$ will denote the other. Hence, by the conditions of the question,

$$x^2+(15-x)^2=113;$$

whence $x=7$, or 8;

and therefore $15-x=8$, or 7.

Thus the two numbers are 7 and 8.

226. In many cases, however, both roots will not satisfy the conditions of the problem. Whenever, for example, the unknown quantity is denoted by $+x$, and one of the roots is negative, this root will be incompatible with the conditions of the given problem. In the following example the unknown time is denoted by $+x$, and therefore the number of gallons that flow through the two cocks in the same direction must have the same sign.

A cistern can be filled in 56 minutes by two cocks flowing together. If they flow separately, it will take one of these cocks an hour and six minutes longer to fill the cistern than the other. In what time will the cistern be filled by each ?

Let the cistern be filled by one cock in x minutes; then it will be filled by the other in $x+66$ minutes.

Also, let g be the number of gallons in the cistern; then the amounts in gallons that flow through the first and second cocks in one minute will be $\frac{g}{x}$ and $\frac{g}{x+66}$, respectively; and $\frac{56g}{x}$, $\frac{56g}{x+66}$ will be the amounts that flow through in 56 minutes.

Hence $$\frac{56g}{x}+\frac{56g}{x+66}=g;$$

$$\frac{56}{x}+\frac{56}{x+66}=1;$$

whence $x=88$, or -42.

The negative root must be rejected as being inconsistent with the conditions of the question; for if $x=-42$, it would follow that the number of gallons flowing through the two cocks in one minute would be $-\frac{g}{42}$ and $+\frac{g}{24}$, respectively.

227. The existence of a positive and a negative root may often be explained by the fact that the latter is the solution of an allied problem which has the same quadratic statement as the given one. In order, however, that the expression of the conditions of these two problems may lead to the same quadratic statement, the unknowns must be so related to each other that they may properly be denoted by $+x$ and $-x$. In the two following examples the unknowns, the number of oxen *bought*, and the number *sold*, may properly be denoted by symbols with opposite signs.

Ex. 1. A person bought a number of oxen for £120, and found that if he had bought 3 more with the same money, he would have paid £2 less for each. How many oxen did he buy?

Let $+x$ denote the number bought; then $\frac{120}{x}$ is the price paid for each, and is positive; if he had purchased 3 more, the price of each would have been $\frac{120}{x+3}$.

Therefore, by the conditions of the problem,

$$\frac{120}{x}-\frac{120}{x+3}=2,$$

which reduces to

$$x^2+3x=180 \quad . \; . \; . \; . \; . \; . \quad (1)$$

Ex. 2. A person sold a number of oxen for £120, and if he had sold 3 fewer for the same money, he would have received £2 more for each. How many oxen did he sell?

Let $-x$ denote the number sold; then the price received for each $=\frac{120}{-x}=-\frac{120}{x}$, a negative quantity. If 3 fewer had been sold, the number disposed of would $=-x+3$, and the price of each would then $=\frac{120}{-x+3}$ $=-\frac{120}{x-3}$.

Therefore, by the conditions of the problem, remembering that money received is in this case negative, we have

$$-\frac{120}{x}-2=-\frac{120}{x-3},$$

which reduces to

$$x^2-3x=180,$$

that is, $$(-x)^2+3(-x)=180 \quad . \; . \; . \; . \; (2).$$

If now we write y for the unknowns of both problems, that is, if we write y for $+x$ in (1) and for $-x$ in (2), those equations reduce to the same form

$$y^2+3y=180 \quad . \; . \; . \; . \; (3)$$

and therefore the two problems have the same quadratic statement (3), the roots of which are 12 and -15, the former of which is the solution of Ex. 1, and the latter of Ex. 2.

Exercises, LVI.

1. Find two numbers such that their sum may be 14, and the sum of their squares 100.

2. Find two numbers such that their sum may be 10, and the sum of their cubes 280.

3. A sum of money, amounting to £10 16s., was divided equally amongst a certain number of persons; if there had been three more, each would have received one shilling less. Find the number of persons.

4. The difference of two numbers is $4\frac{1}{2}$, and their product 28. Find the numbers.

5. The product of two numbers is 36, and the difference of their squares 65; find the numbers.

6. A gentleman bought a horse for a certain sum, and having resold it for £119, found that he had gained as much per cent. as the horse cost him. What was the prime cost of the horse?

7. A, working alone, can perform a piece of work in 10 days less than B takes to perform it alone; both together can perform the work in 12 days: how long does it take A to do it alone?

8. A and B, working together, can perform a piece of work in 10 days; after working together for 4 days, A is taken ill, and B finishes the work in 3 days more than A would have taken to do the whole: in what time would each of them do it separately?

9. Find two numbers whose sum is 100, and the difference of their square roots 2.

10. The height of a certain triangle is 4 inches less than the base; if the base be increased 6 inches and the height lessened as much, the area is diminished by one-eighth part: find the base of the triangle?

11. A rectangular field is an acre in extent, and its perimeter is 308 yards: what are the lengths of its sides?

12. A boat's crew can row in still water at the rate of 6 miles an-hour. They enter a current, and row

with it for a distance of 5 miles; on coming back they find that it takes them two hours longer to make the same distance against the current than the time it took them when rowing with it. At what rate did the current run?

13. Two vessels, one of which sails faster than the other by 2 miles an-hour, start at the same time upon voyages of 1152 and 720 miles, respectively; the slower vessel reaches its destination one day before the other: how many miles did the faster vessel sail?

14. A number is composed of two digits, the first of which exceeds the second by unity, and the number itself falls short of the sum of the squares of its digits by 26. What is the number?

15. A number is composed of two digits, the first of which exceeds the second by 2. The sum of the squares of the number, and of that which is obtained by reversing the digits, is 4034. What is the number?

16. A merchant sells two casks of wine for £76 5s.; one holds 5 gallons more than the other, and the price of each wine is in shillings the number of gallons in the cask which contains it. How many gallons are there in each cask?

17. A person drew a quantity of wine from a full vessel which held 81 gallons, and then filled up the vessel with water. He then drew from the mixture as much as he before drew of pure wine; and it was found that 64 gallons of pure wine remained. Find how much he drew each time.

18. In order to resist cavalry, a battalion is usually formed into a hollow square, the men being four deep, but a single company is usually formed into a solid square. If the hollow of the square of a battalion, consisting of seven equal companies, is nine times as

large as one of its companies' squares, find how many men there are in a company, assuming every man to occupy the same amount of space.

PROPERTIES OF QUADRATIC EQUATIONS.

228. The solution of the equation $x^2+px+q=0$ gives the two roots in terms of p and q. Let them be denoted by a and b, so that

$$a=-\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q},$$

$$b=-\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q}.$$

229. From these values of a and b we can deduce other relations between a, b, p, q.

Thus, for example, if we separately add and multiply together the values of a and b, we obtain

$$a+b=-p,\ a\,b=q.$$

Hence the sum of the roots of $x^2+px+q=0$ is equal to the coefficient of x with its sign changed, and the product of the roots is equal to the absolute term q.

Thus in the equation $x^2+5x-3=0$, the sum of the roots is -5, and their product -3.

230. If the given equation be $ax^2+bx+c=0$, or $x^2+\frac{b}{a}x+\frac{c}{a}=0$, the sum of the roots will be $-\frac{b}{a}$, and their product $\frac{c}{a}$.

Thus in the equation $5x^2-9x-10=0$, the sum of the roots is $\frac{9}{5}$ and their product -2.

231. If the equation be given in the form $(x-a)(x-b)=0$, the roots of which are a and b, the foregoing relation is evident, for the equation becomes, on multiplying the factors of the left-hand side,

$$x^2-(a+b)x+ab=0,$$

in which the coefficient of x with its sign changed is $a+b$, the sum of the roots, and the absolute term ab is their product.

232. Hence the equation $x^2+px+q=0$ can always be written in the form

$$x^2-(a+b)x+ab=0,$$

$$\text{or, } (x-a)(x-b)=0.$$

233. Hence also every expression x^2+px+q can be resolved into two factors of the first degree in x.

Thus, for example, $x^2-11x+18=(x-2)(x-9)$, where 2 and 9 are the roots of $x^2-11x+18=0$.

234. An expression of the form ax^2+bx+c can be resolved in like manner, by putting it into the form

$$a\left(x^2+\frac{b}{a}.x+\frac{c}{a}\right)$$

the second factor of which is resolved as before.

Thus, for example, $6x^2+5x-4=6\left(x^2+\frac{5}{6}x-\frac{2}{3}\right)=6\left(x-\frac{1}{2}\right)\left(x+\frac{4}{3}\right)$, where $\frac{1}{2}$ and $-\frac{4}{3}$ are the roots of $x^2+\frac{5}{6}x-\frac{2}{3}=0$.

235. Conversely, if the roots be given, the equation may be formed.

Thus, the equation whose roots are 5 and -6, is

$$x^2-(5-6)x-30=0,$$

$$\text{or } x^2+x-30=0;$$

and the equation whose roots are $-\frac{4}{3}$ and $\frac{5}{6}$, is

$$x^2-\left(\tfrac{5}{6}-\tfrac{4}{3}\right)x-\tfrac{4}{3}\cdot\tfrac{5}{6}=0,$$

$$\text{or, } 18x^2+9x-20=0.$$

Exercises, LVII.

Resolve into simple factors

1. $x^2-11x+18$.
2. $x^2-9x-52$.
3. $6x^2+5x-4$.
4. $7-9x-10x^2$.
5. $33-14x-40x^2$.
6. $209x^2-302x+65$.

7. Shew that in every quadratic of the form $ax^2+bx+a=0$, the roots are the reciprocals of each other.

8. Shew that the roots of $cx^2+bx+a=0$, are the reciprocals of those of $ax^2+bx+c=0$.

236. The relations proved in Art. 229 may be employed to establish other relations connecting the coefficients and the roots.

Thus, since $a+b=-p$, $ab=q$, if we divide the members of the former equation by those of the latter, we obtain

$$\frac{a+b}{ab}=-\frac{p}{q},$$

$$\text{or, } \frac{1}{a}+\frac{1}{b}=-\frac{p}{q}.$$

Hence the sum of the reciprocals of the roots of $x^2+px+q=0$ is $-\frac{p}{q}$.

The sum of the squares of the roots may be obtained as follows:

$$a^2+b^2=(a+b)^2-2ab=(-p)^2-2q=p^2-2q.$$

237. The nature of the roots $-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}$ of $x^2+px+q=0$ is shewn, as follows, to depend on the nature of the quantity $\sqrt{p^2-4q}$.

238. I. If $p^2-4q=0$, or $p^2=4q$, it follows that $a=b=-\frac{p}{2}$, and therefore the roots are equal, as is otherwise evident, since the left-hand member of the equation is then reduced to the form $x^2+px+\frac{p^2}{4}=\left(x+\frac{p}{2}\right)^2$.

239. II. If p^2-4q be positive, that is, if $p^2>4q$, the roots will be real and different, a being the sum, and b the difference, of the same two real quantities $-\frac{p}{2}, \frac{1}{2}\sqrt{p^2-4q}$. This will always be the case if the absolute term be negative.

For example, let the absolute term be $-r$, then the quantity under the radical sign is p^2+4r, which is positive, and therefore both roots are real.

Thus, for example, the roots of $x^2+5x+2=0$ are real and different, because $5^2>4\times 2$; the roots of $x^2-7x+8=0$ are real and different, because $(-7)^2>4\times 8$; and the roots of $x^2+7x-1=0$, $x^2-10x-3=0$, are seen by inspection to be real and different, because their absolute terms are negative.

240. III. If p^2-4q be negative, that is, if $p^2<4q$, the roots will be imaginary, because the square

root of a negative quantity will then enter into the value of each root.

Thus the roots of $x^2-5x+7=0$, are imaginary, because $(-5)^2<4\times7$, or $25<28$.

241. It follows, therefore, that the roots of $x^2+px+q=0$ are real and unequal, real and equal, or imaginary, according as p^2 is greater than, equal to, or less than $4q$, that is, according as p^2-4q is greater than, equal to, or less than zero.

242. If the quadratic be of the form $ax^2+bx+c=0$, its roots will depend in like manner on $\sqrt{b^2-4ac}$, which enters into the value of each. Hence its roots will be real and unequal, real and equal, or imaginary, according as

$$b^2-4ac \begin{matrix} > \\ = \\ < \end{matrix} 0.$$

243. The preceding tests for determining the nature of the roots of a quadratic may sometimes be employed to find the *maximum* and *minimum* values of a function of x.

244. A function of x is said to attain its maximum or minimum value when by assigning gradually increasing or diminishing values to x, the function ceases to increase, and begins to diminish, or *vice versâ*.

Thus, the quantity $x^2-2x+5=x^2-2x+1+4=(x-1)^2+4$, has a minimum value 4 corresponding to

$x=1$: for all other real values of x render $(x-1)^2$ positive, and, therefore, $(x-1)^2+4$ greater than 4.

245. In like manner it follows that the quantity $x^2+px+q=(x+\frac{p}{2})^2+\frac{4q-p^2}{4}$, has a minimum value $\frac{4q-p^2}{4}$, corresponding to the value $x=-\frac{p}{2}$.

Exercises, LVIII.

Find the minimum value of

1. x^2-3x+5. 2. x^2+4x+8. 3. $x^2-\frac{2}{3}x+\frac{1}{6}$.

4. $50x^2+40x+9$. 5. $72x^2-63x+64$.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE CONTAINING TWO UNKNOWN QUANTITIES.

246. An equation containing two unknowns is said to be of the degree indicated by the highest total dimensions of the unknowns involved in any term.

Thus $2x-3y=5$ is an equation of the first degree in x and y; $x^2-y^2=2$, $xy=8$, $x+2xy=3$ are equations of the second degree; and so on.

247. If there be only one such equation between x and y the number of solutions will be infinite, for the equation will be satisfied by giving any value to

one unknown, when the other unknown will have one or more corresponding values.

Thus in the equation $2x-3y=5$, if we give y the value 1, $x=4$; if $y=3$, $x=7$; if $x=3$, $y=\frac{1}{3}$; and so on; so that the equation is satisfied by the sets of values 4, 1; 7, 3; 3, $\frac{1}{3}$; &c.

Again, the equation $x^2-y^2=2$ is satisfied by giving any value to one unknown, when the other unknown has two corresponding values.

Thus, if $y=2$, $x=\pm 2$; if $y=1$, $x=\pm\sqrt{3}$.

248. If there be two such independent equations in x and y at the same time, they are called *simultaneous* equations, the same values of the unknowns being taken in both equations. In such a case the number of solutions is finite.

Thus, if we have the simultaneous equations $2x-3y=5$, $3x+y=13$, it will be found that the only set of values which will satisfy these equations is $x=4$, $y=1$.

Again, the equations $2x-y=4$, $x^2-2xy+3y^2=9$, the former of which is of the first, and the latter of the second degree, will be found to have the two solutions $x=3$, $y=2$; $x=\dfrac{13}{9}$, $y=-\dfrac{10}{9}$; either of which sets of values, and no others, will satisfy the equations.

249. And, generally, it may be stated (as is proved in works on the *Theory of Equations*) that if the two equations in x and y be of the mth and nth degrees, respectively, the number of solutions may be equal to, but not greater than, mn.

Thus the number of solutions of the equations $xy+y^2=6$, $x^2-3xy+2y^2=3$, which are of the second and third degrees, respectively, cannot be greater than 2×3, or 6.

250. It must be carefully observed that the number of solutions will not be finite unless the two simultaneous equations be independent. If, for instance, one equation follows necessarily from the other, the two are really equivalent to one, and therefore the number of solutions will be infinite.

For example, the equations $2x-3y=1$, $8x-12y=4$, have an infinite number of solutions, for the second equation is deduced by multiplying the sides of the first by 4, and therefore there is only one independent equation.

Again, the equations

$$x+2y-1=0,$$
$$2x^2+3xy-2y^2+x+7y-3=0,$$

have an infinite number of solutions, for the second is deduced from the first by multiplying its sides by $2x-y+3$.

251. The solution of such simultaneous equations is effected by deducing from them two other equations involving each one unknown quantity. This process is called *elimination*, and may be conducted according to one of the following methods:

I. Substitution.

II. Comparison.

III. Cross Multiplication.

IV. Arbitrary Multiplier.

These methods will be illustrated, in the first instance, in solving simple equations of two unknowns.

METHOD OF SUBSTITUTION.

252. This method consists in finding from one equation the value of one unknown in terms of the other, and substituting the value so found in the second equation, which is thence reduced to a simple equation in one unknown.

For convenience of reference the given equations and others which arise in the process of solution are numbered (1), (2), (3), &c.

EXAMPLE.

Solve $x+y=3$............(1),

$2x+y=4$............(2).

From (1) we find $y=3-x$ ......(3).

Substituting this value of y in (2) we obtain

$$2x+3-x=4,$$

$$x=1.$$

This value of x substituted in (3) gives $y=2$.
Hence the solution is $x=1$, $y=2$.

EXERCISES, LIX.

1. $3x-4y=2,\ 7x-9y=7$.
2. $5x+2y=29,\ 6x-2y=26$.
3. $3x+2y=0,\ 3x-4y=18$.
4. $5x+4=2y,\ 4x+y=28$.
5. $2x-3y=0,\ 4x-\dfrac{y}{2}=11$.
6. $6x-5y=1,\ 7x-4y=8\frac{1}{2}$.
7. $2x+5y=2\frac{1}{2},\ 5x+2y=2\frac{1}{20}$.

8. $\frac{3x}{19}+5y=13,\ 2x+\frac{4-7y}{2}=33.$

9. $\frac{x+y}{3}+\frac{y-x}{2}=9,\ \frac{x}{2}+\frac{x+y}{9}=5.$

10. $4x+5y=40(x-y),\ 2x+5y=1\frac{1}{2}.$

11. $bx-ay=0,\ \frac{x}{a}+\frac{y}{b}=2.$

12. $\frac{x}{a}+\frac{y}{b}=1,\ \frac{x}{b}+\frac{y}{a}=1.$

METHOD OF COMPARISON.

253. This method consists in finding from each of the proposed equations the value of one and the same unknown quantity in terms of the other, and equating the values so found.

EXAMPLE.

Solve $7x-3y=19 \ldots\ldots\ldots\ldots(1),$

$4x+7y=37 \ldots\ldots\ldots\ldots(2).$

From (1) we obtain

$$y=\frac{7x-19}{3}\ldots\ldots\ldots\ldots(3);$$

and from (2) $y=\frac{37-4x}{7}\ldots\ldots\ldots\ldots(4).$

Equating these values of y we get a simple equation in x,

$$\frac{7x-19}{3}=\frac{37-4x}{7};$$

whence $x=4.$

Substitute this value of x in (3) or (4), and we get $y=3$.

Thus the solution is $x=4$, $y=3$.

EXERCISES, LX.

1. $5x+11y=146,\ 11x+5y=110.$

2. $3x-2y=19,\ 2x+3y=43.$

3. $7x-2y=14+\frac{x}{2},\ 7y-2x=32+\frac{y}{3}.$

4. $\frac{3x+5y}{20}+\frac{5x-3y}{8}=3,\ \frac{x+1}{y+2}=\frac{2}{3}.$

5. $\frac{bx+ay}{a}=c=\frac{ax-by}{b}.$

6. $\frac{7x+6}{11}+y-16=\frac{5x-13}{2}-\frac{8y-x}{5},$
$3(3x+4)=10y-15.$

7. $ax+by=c,\ px+qy=r.$

8. $\frac{x+1}{y+a}=\frac{x-1}{y+b},\ \frac{x+a}{y+1}=\frac{x+b}{y-1}.$

9. $ax+by=c,\ a^2x+b^2y=c^2.$

METHOD OF CROSS MULTIPLICATION.

254. This method consists in multiplying the given equations (reduced to their simplest forms) by such quantities as will render the coefficients of the same unknown numerically equal. By adding or subtracting the equations so found we obtain a simple equation in one unknown.

EXAMPLES.

1. Solve $7x-9y=29$ ....(1),
$13x+4y=116$ ....(2).

Multiply (1) by 4 and (2) by 9; thus

$28x-36y=116$.........(3),
$117x+36y=1044$ ......(4),

by adding (3) and (4) we eliminate y;

$$145x=1160,$$

$$x=8.$$

To eliminate x, multiply (1) by 13 and (2) by 7,

$$91x-117y=377 \ldots\ldots\ldots(5),$$

$$91x+28y=812 \ldots\ldots\ldots(6);$$

subtract (5) from (6),

$$145y=435,$$

$$y=3.$$

Thus the roots are 8, 3.

2. Solve $15x+4y=54 \ldots\ldots\ldots\ldots(1),$

$$13x-6y=61 \ldots\ldots\ldots\ldots(2).$$

Here, in order to eliminate y, we may multiply (1) by 3, and (2) by 2, 3 and 2 bearing the same proportion to each other as 6 and 4; thus

$$45x+12y=162 \ldots\ldots\ldots\ldots(3),$$

$$26x-12y=122 \ldots\ldots\ldots\ldots(4).$$

Add (3) and (4),

$$71x=284,$$

$$x=4.$$

To eliminate x, multiply (1) by 13, and (2) by 15, and subtract the equations so found,

$$142y=-213,$$

$$y=-\frac{3}{2}$$

3. Solve $\frac{1}{4x}-\frac{1}{3y}=\frac{7}{6} \ldots\ldots\ldots\ldots(1),$

$$\frac{1}{x}-\frac{1}{y}=\frac{23}{6} \ldots\ldots\ldots(2),$$

by first finding $\frac{1}{x}, \frac{1}{y}$.

Here, in order to eliminate y, it is only necessary to divide (2) by 3,

$$\frac{1}{3x}-\frac{1}{3y}=\frac{23}{18};$$

and subtract from this (1),

$$\frac{1}{3x}-\frac{1}{4x}=\frac{23}{18}-\frac{7}{6},$$

$$x=\frac{3}{4}.$$

To eliminate x divide (2) by 4, and [illegible]ct from (1); whence we obtain $y=-\frac{2}{5}$.

255. Instead of using either of the proposed equations we may use any equation derived from them which has smaller numerical coefficients, as in the following

EXAMPLE.

Solve $$16x-15y=-38\ldots\ldots\ldots(1),$$
$$12x-13y=-46\ldots\ldots\ldots(2).$$

Subtracting (2) from (1), we obtain

$$4x-2y=8\ldots\ldots\ldots\ldots$$

$$\text{or, }\ 2x-\ \ y=4\ldots\ldots\ldots\ldots(3).$$

Equation (3) may be used instead of either of the proposed equations, as (1); thus we have to determine x and y from the equations

$$12x-13y=-46,$$
$$2x-y\ \ =\ \ 4.$$

Eliminating as before we obtain $x=7$, $y=10$.

EXERCISES, LXI.

1. $5x+2y=12$, $3x+5y=11$.
2. $8x-7y=11$, $5x-3y=11$.

3. $9x-7y=7,\ 4x+3y=60.$

4. $\frac{x}{4}-\frac{y}{2}=1,\ \frac{x}{12}+\frac{y}{6}=1.$

5. $32x+81y=43,\ 28x-39y=1.$

6. $96x+75y=102,\ 92x+80y=101.$

7. $x+y=a+b,\ bx+ay=2ab.$

8. $x+y=c,\ ax-by=c(a-b).$

9. $a(x+y)+b(x-y)=1,\ a(x-y)+b(x+y)=1.$

10. $a(x-a)+b(y-b)=0,\ a(x-y-a)+b(x+y-b)=0.$

METHOD OF ARBITRARY MULTIPLIER.

256. This method consists in multiplying either of the proposed equations by an arbitrary multiplier m, then adding the other equation to the equation so found, equating to zero the coefficient of y and solving for x, or the coefficient of x and solving for y; the value of m corresponding to each case being found from the coefficient which is equated to zero.

EXAMPLE.

Solve $4x+5y=7$ ....(1),

$3x-10y=19$............(2).

Multiply (2) by m, and add to (1); thus

$$(4+3m)x+(5-10m)y=7+19m\text{.........}(3).$$

Equation (3) is true for all values of m, and therefore for that value which makes $5-10m=0$, or $m=\frac{1}{2}$; in which case (3) becomes

$$\left(4+\frac{3}{2}\right)x=7+\frac{19}{2},$$

$$x=3.$$

Equation (3) is also true for that value of m which makes $4+3m=0$, or $m=-\frac{4}{3}$, in which case (3) becomes

$$\left(5+\frac{40}{3}\right)y=7-\frac{76}{3},$$

$$y=-1.$$

EXERCISES, LXII.

1. $15x+28y=58,\ 11x-7y=15.$
2. $7x+3y=13,\ 21x-6y=114.$
3. $8x+3y=5,\ 10x+9y=8.$
4. $2x+y=\frac{1}{2},\ 18x+22y=-28.$
5. $11x+9y=40,\ \frac{3x+y}{8}-\frac{2x-y}{31}=2.$
6. $ax+by=c,\ mx+ny=p.$
7. $rx+m=ty+n,\ \frac{x+y}{x-y}=\frac{m}{n}.$
8. $\frac{m}{n+y}=\frac{n}{m-x},\ \frac{p}{q-x}=\frac{q}{p+y}.$

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE CONTAINING THREE UNKNOWNS.

257. When three independent equations containing three unknowns are given, the number of sets of solutions depends on the degree of each equation, and (as is proved in the *Theory of Equations*) may be equal to but not greater than the product of the numbers expressing the degrees of the three equations.

Thus, if each equation be of the first degree, there will be only one solution; if the first equation be of the first degree, the second equation of the second degree and the third equation of the third degree, the number of solutions may be equal to, but not greater than, $1 \times 2 \times 3$, or 6; and, generally, if the first equation be of the mth, the second equation of the nth, and the third equation of the pth degree, the number of solutions may equal, but not exceed, mnp.

If the three equations be not independent, that is, if one of them be deducible from the other two, or if two of them be deducible from the remaining one, it will follow, as in the case of two unknowns, Art. 250, that the number of solutions will be infinite.

258. If the three given equations are of the first degree, the values of the unknowns may be found by eliminating one of them between the first and second equations, and also between the first and third, or second and third; whence we shall obtain two equations involving only two unknowns, which may be found by previous methods. The value of the third may be found by substitution.

EXAMPLE.

Solve $2x+3y+4z=20$............(1),

$4x-3y-2z=-8$ ....(2),

$3x+4y+5z=26$............(3).

Here z is eliminated from (1) and (2) by adding (1) to twice (2); thus,

$10x-3y=4$ ....(4).

Also z is eliminated from (2) and (3) by adding five times (2) to twice (3); thus

$26x-7y=12$ ....(5).

From (4) and (5) we obtain $x=1$, $y=2$.

Substitute these values in any one of the proposed equations, and we obtain $z=3$.

Thus the solution is $x=1$, $y=2$, $z=3$.

259. By the following method of Cross Multiplication we can eliminate two of the unknowns at the same time by multiplying the given equations by certain quantities, and adding the derived equations.

Let the given equations be

$$a_1x+b_1y+c_1z+d_1=0 \ldots\ldots(1),$$

$$a_2x+b_2y+c_2z+d_2=0 \ldots\ldots(2),$$

$$a_3x+b_3y+c_3z+d_3=0 \ldots\ldots(3).$$

To eliminate y and z multiply (1) by $b_2c_3-b_3c_2$, (2) by $b_3c_1-b_1c_3$, and (3) by $b_1c_2-b_2c_1$, and add the equations so found. The coefficients of y and z in the resulting equation vanish, and we obtain

$$\{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)\}\,x$$
$$+d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1)=0,$$

whence x is found.

To eliminate z and x the system of multipliers for (1), (2), (3) will be $a_2c_3-a_3c_2$, $a_3c_1-a_1c_3$, $a_1c_2-a_2c_1$; and to eliminate x and y they will be $a_2b_3-a_3b_2$, $a_3b_1-a_1b_3$, $a_1b_2-a_2b_1$.

260. When the proposed equations have numerical coefficients, it will be found convenient to obtain the numerical values of the multipliers and determine their proper signs by trial.

EXAMPLE.

Solve

$$2x-3y+z=1 \;.\;.\;.\;.\;.\;.\; (1),$$

$$3x-5y+4z=3 \;.\;.\;.\;.\;.\;.\; (2),$$

$$4x+2y-3z=13 \;.\;.\;.\;.\;.\; (3).$$

To eliminate y and z, the multipliers for the three equations are formed from the second and third columns, and are $-5\ (-3)-2\times 4$, $-3\ (-3)-2\times 1$, $-3\times 4-\ (-5)\times 1$, whose values are, without regard to sign 7, 7, 7. These may be replaced by the three numbers in the same ratio, 1, 1, 1. To determine the proper signs, we apply these multipliers to the coefficients of y or z, and add, when the sum should be zero. Here, taking the coefficients of y, we evidently have $-1\ (-3)+1\ (-5)+1\times 2=0$, and therefore the proper multipliers are -1, 1, 1.

Multiply (1), (2), and (3), therefore, by -1, 1, 1, respectively, and add. The coefficients of y and z in the resulting equation vanish, and we obtain

$$5x=15,$$

$$x=\ 3.$$

To eliminate x and z, the multipliers are formed from the first and third columns, and are, without regard to sign, 25, 10, 5, which may be replaced by 5, 2, 1, in the same ratio. Apply these to the coefficients in the first column, and evidently the first multiplier should be -5, in order that the sum of the products should be zero. Multiply (1), (2), (3), therefore, by -5, 2, 1, and we obtain

$$7y=14,$$

$$y=\ 2.$$

Finally to eliminate x and y, the proper multipliers are found from the first and second columns to be 26, 16, 1. Applying these to the three equations, we obtain

$$35z=35,$$

$$z=\ 1.$$

Hence the solution is $x=3$, $y=2$, $z=1$.

EXERCISES, LXIII.

1. $x+3y+2z=11,\ 2x+y+3z=14,\ 3x+2y+z=11.$

2. $5x-6y+4z=15,\ 7x+4y-3z=19,$
$2x+y+6z=46.$

3. $4x-5y+z=6,\ 7x-11y+2z=9,\ x+y+3z=12.$

4. $7x-3y=30,\ 9y-5z=34,\ x+y+z=33.$

5. $3x-y+z=17,\ 5x+3y-2z=10,\ 7x+4y-5z=3.$

6. $x+y+z=5,\ 3x-5y+7z=75,\ 9x-11z+10=0$

7. $\frac{x}{2}+\frac{y}{4}+\frac{z}{6}=3,\ \frac{x}{4}+y-\frac{z}{12}=4,\ \frac{x}{5}+y+\frac{z}{10}=5.$

8. $\frac{x}{5}-\frac{y-z}{2}=2,\ \frac{x+y+z}{7}=3,\ \frac{x+y}{4}+\frac{z}{9}=4.$

9. $y+z=a,\ z+x=b,\ x+y=c.$

10. $x+y+z=a+b+c,\ x+a=y+b=z+c.$

11. $y+z-x=a,\ z+x-y=b,\ x+y-z=c.$

12. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,\ \frac{x}{a}+\frac{y}{c}+\frac{z}{b}=1,\ \frac{x}{b}+\frac{y}{a}+\frac{z}{c}=1.$

PROBLEMS LEADING TO SIMPLE EQUATIONS, WITH TWO OR MORE UNKNOWN QUANTITIES.

261. When there are two or more unknowns in a problem to be determined, and they are not imme-

diately expressible in terms of one unknown, as in the cases referred to in Art. 209, they must be represented by distinct symbols. The conditions of the problem being then translated into Algebraical language will lead to a corresponding number of independent simultaneous equations, the solution of which will give the values sought.

EXAMPLES.

1. What fraction is that which becomes, when its numerator is increased by 7, equal to $\frac{3}{2}$, and when its denominator is increased by 10, equal to $\frac{1}{2}$?

Let x denote the numerator, and y the denominator of the fraction. If the numerator x be increased by 7, the denominator remaining unchanged, the fraction becomes $\frac{x+7}{y}$; and if the denominator y be increased by 10, the numerator remaining unchanged, the fraction becomes $\frac{x}{y+10}$. Hence, by the conditions of the problem,

$$\frac{x+7}{y}=\frac{3}{2},$$

$$\frac{x}{y+10}=\frac{1}{2};$$

the solution of which is $x=11$, $y=12$.

The fraction is, therefore $\frac{11}{12}$.

2. There is a number consisting of two digits, which exceeds five times the digit in the unit's place by 6;

and if 27 be added to it, the sum will be expressed by the same digits in an inverted order.

Let x be the digit in the place of tens, and y the digit in the place of units.

By the first condition of the problem we have

$$10x+y=5y+6;$$

and by the second condition,

$$10x+y+27=10y+x.$$

From these two equations we find $x=3$, $y=6$.

Therefore the number sought is 36.

3. A and B can do a piece of work in 6 days; A and C can do it in 9 days, and A, B, C can do 8 times the same work in 45 days. In what times can they do it separately?

Let x, y, and z denote the numbers of days in which A, B, and C, respectively, can do the work w.

Then, since $\frac{w}{x}$, $\frac{w}{y}$, $\frac{w}{z}$ are the amounts of work done by them separately in one day, we have by the conditions of the problem

$$\frac{w}{x}+\frac{w}{y}=\frac{w}{6}$$

$$\frac{w}{x}+\frac{w}{z}=\frac{w}{9},$$

$$\frac{w}{x}+\frac{w}{y}+\frac{w}{z}=\frac{8w}{45}.$$

These equations become, on dividing by w,

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{6} \quad \ldots\ldots\ldots \quad (1)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{9} \quad \ldots\ldots\ldots \quad (2)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{8}{45} \quad \ldots\ldots \quad (3)$$

By subtracting (1) and (2) from (3), we find $\frac{1}{z}=\frac{1}{90}$, and $\frac{1}{y}=\frac{1}{15}$, the reciprocals of which give $z=90$, and $y=15$. The value of x is then found, by substitution, to be 10.

Exercises, LXIV.

1. Find two numbers, such that the sum of the first and half the second shall be 19, and the sum of the second and half the first shall be 20.

2. A number composed of two digits is equal to seven times its unit's figure; and, if the digits be reversed, its value is increased by 18. Required the number.

3. If the numerator of a certain fraction be increased by 2, its value becomes one-half; but if the numerator be diminished by 2, the value of the fraction is one-fourth. What is the fraction?

4. Three numbers are required whose sum shall be 22; the sum of the first and third to exceed twice the second by unity; and four times the first, three times the second, and twice the third added together, to make 69.

5. Find two numbers such that twice the first and three times the second shall together make 18; and, if double the second be taken from five times the first, 7 shall remain.

6. A cistern supplied by two taps is filled by both together in 50 minutes; after both had been open together for 35 minutes, one is closed, and the cistern is found to be full 27 minutes afterwards. In what time would each tap fill the cistern?

7. A man being asked what money he possessed replied that he had three sorts of coin, namely, half-crowns, shillings, and sixpences; the shillings and sixpences together amounted to 409 pieces, the shillings and half-crowns to 1,254 pieces; but if 42 was subtracted from the sum of the half-crowns and sixpences, there would remain 1,103 pieces. What sum did the man possess in all?

8. A courier travelled a distance of 240 miles in four days, diminishing his rate of travelling each day alike; during the first two days he travelled 136 miles. What number of miles did he travel on each day respectively?

9. A and B can do a piece of work in 2 days; A and C can do four times as much in 9 days; A, B, and C can do eleven times as much in 18 days. In how many days can each do it separately?

10. The ten's digit of a number is less by 2 than the unit's digit; and, if the digits are inverted, the new number is to the former as 7 to 4. Find the digits.

11. The sum of three numbers is $(p+1)(q+1)n$; the sum of the two larger is equal to p times the smallest, and the sum of the two smaller to q times the largest. Find the numbers.

12. At an examination there were 17 candidates, of whom some were passed, some sent back in one subject, and the rest rejected. If one less had been rejected, and one less sent back, the number of those passed would have been twice those rejected, and five times those sent back. How many of each class were there ?

13. There are two numbers in the proportion 3 : 5 ; if 10 be added to the first, and subtracted from the second, the proportion is reversed. Find the numbers.

14. Find three numbers, such, that one-half the first, one-third the second, and one-fourth the third, together shall be equal to 62 ; one-third of the first, one-fourth of the second, and one-fifth of the third equal to 47 ; and, finally, one-fourth of the first, one-fifth of the second, and one-sixth of the third equal to 38.

15. A man has two casks, each containing a quantity of wine. From the first he pours into the second as much as is already there ; from the second he pours back into the first as much as is already there ; and, finally, from the first he pours into the second as much as is already there : he then finds there are 80 quarts in each. How many were in each originally ?

16. A number consists of two digits whose sum is 12, and such that, if the digits be reversed in order, the number produced will be less by 36. Find the number.

17. A, B, and C sat down to play, and found when they stopped that they each had the same amount of money. When they commenced, A had £11 8s. less than B and C together ; he now finds he has lost £21 4s., that B and C have won £18 14s. and £2 10s., respectively. With what money did each begin, and with how much did they conclude the game ?

18. A cistern can be filled from three different cocks. If the first and second run together, it will be filled in $33\frac{1}{3}$ minutes; if the first and third run together, in 36 minutes; if the second and third run together, in $40\frac{10}{11}$ minutes. How long will it take for each separately, and for all running together?

19. A silversmith has two alloys of silver; he melts 10 ounces of the first with 5 ounces of the second, and produces an alloy, the fineness of which is $687\frac{1}{2}$ per mille; again, he melts $7\frac{1}{2}$ ounces of the first with $1\frac{1}{2}$ ounces of the second, and produces an alloy which is 625 per mille fine. What is the fineness of each kind?

20. Three towns, A, B, and C, are at the angles of a triangle. From A to C, through B, the distance is 82 miles; from B to A through C, is 97 miles; and from C to B, through A, is 89 miles. Find the direct distances through the towns.

21. The diameter of a five-franc piece is 37 *millimètres*, and of a two-franc piece is 27 *millimètres*. Thirty pieces laid in contact in a straight line measure one *mètre* exactly. How many of each kind are there?

22. A man invested £2,000 in cottages; a certain number were then burnt down; but, in consequence of their having been insured, the loss in each cottage was only 10 per cent. on its cost price, and was also found to be at the rate of £10 on each cottage bought; some time after the remainder were sold 20 per cent. dearer than they had been bought, and the total gain was £100. How many cottages were originally bought, and how many were burnt?

SIMULTANEOUS EQUATIONS, WHEREOF ONE IS OF THE FIRST DEGREE AND THE OTHER OF THE SECOND DEGREE IN TWO UNKNOWNS.

262. Let the general equation of the second degree be

$$ax^2+bxy+cy^2+dx+ey+f=0\ldots\ldots(1),$$

and of the first degree

$$a'x+b'y=c'\ldots\ldots\ldots\ldots(2).$$

In order to solve these equations we substitute in (1) for one unknown, as y, its value in terms of x derived from (2); (1) thus becomes a quadratic in x, and has two roots. By substituting these values of x in (2) we get two corresponding values for y.

Example.

Solve $3x^2-6xy+4y^2-2x-2y+2=0\ldots\ldots(1)$,

$$6x-2y=10\ \ldots\ldots\ldots\ldots(2).$$

From (2) we find $y=3x-5\ \ldots\ldots\ldots\ldots\ldots(3)$;

substituting this value for y in (1), we obtain

$$21x^2-98x+112=0,$$

the roots of which are $\frac{8}{3}$ and 2.

Substituting these values for x in (3), we find the corresponding values of y to be 3 and 1.

The required solutions are, therefore, $x=\frac{8}{3}$, $y=3$; and $x=2$, $y=1$.

263. For certain forms of the general equation of the second degree in x and y the preceding method

may be advantageously replaced by others. The various artifices are exhibited in the following

EXAMPLES.

1. Solve $x+y=14 \ldots\ldots\ldots\ldots\ldots\ldots (1),$

$$xy=13 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In this case, squaring (1) and subtracting 4 times (2), we get

$$x^2-2xy+y^2=144;$$

extract the square root

$$x-y=\pm 12 \ldots\ldots\ldots (3).$$

Combining (1) and (3) by addition and subtraction we obtain, if $+12$ be taken in (3), $x=13$, $y=1$; and, if -12 be taken, $x=1$, $y=13$.

The same method will answer when the first member of the first equation is $x-y$, or $mx+ny$. In the latter case (2) is multiplied by $4mn$, and then subtracted from the square of (1).

2. Solve $8x^2-22xy+15y^2=5 \ldots\ldots\ldots (1),$

$$2x-3y=1 \ldots\ldots\ldots (2).$$

Here (1) is equivalent to

$$(2x-3y)(4x-5y)=5 \ldots\ldots\ldots (3);$$

substituting in (3) for $2x-3y$ its value from (2), (3) becomes

$$4x-5y=5 \ldots\ldots\ldots\ldots (4).$$

From (2) and (4) we find $x=5$, $y=3$.

There is, therefore, only one solution in this case. That this must be so may also be seen by substituting in (1) for one unknown its value from (2), when (1) is reduced to a simple equation in one unknown.

3. Solve $4x^2+12xy+9y^2=64\ldots\ldots\ldots(1),$

$$3x-y=1\ \ldots\ldots\ldots(2).$$

Here the left hand side of (1) is a perfect square;

$$\text{therefore } 2x+3y=\pm 8\ldots\ldots\ldots(3).$$

Combining (2) and (3), we find $x=1$, $y=2$; or $x=-\frac{5}{11}$, $y=-\frac{26}{11}$.

4. Solve $x^2-2xy+y^2=9\ldots\ldots\ldots\ldots(1),$

$$2x-y=4\ldots\ldots\ldots\ldots(2).$$

From these two equations we may derive an equation whose first member shall be a perfect square, as follows:

Multiply (1) by m, and add to square of (2); thus

$$(m+4)x^2-2(m+2)xy+(3m+1)y^2=9m+16\ldots(3).$$

In order that the left-hand member of (3) may be a perfect square we must have, by Art. 242,

$$(m+2)^2=(m+4)(3m+1),$$

$$\text{or, } m=-\frac{9}{2}.$$

If, therefore, (1) be multiplied by $-\frac{9}{2}$, and added to the square of (2), we will get

$$x^2-10xy+25y^2=49,$$

the left-hand member of which is a perfect square;

$$\text{therefore } x-5y=\pm 7\ldots\ldots\ldots\ldots(4).$$

Combining (2) and (4) we find $x=\frac{13}{9}$, $y=-\frac{10}{9}$ or $x=3$, $y=2$.

5. Solve $xy+1=4y$...............(1),

$x=4y$...............(2).

Divide (1) by y, and write u for $\frac{1}{y}$; (1) and (2) may then be written

$$x+u=4,$$
$$xu=4.$$

These may now be solved by the method of Ex. 1.

EXERCISES, LXV.

1. $x+y=12$, $xy=11$.
2. $x-y=1$, $xy=600$.
3. $x^2+y^2=20008$, $xy=9996$.
4. $3x+2y=14$, $xy=4$.
5. $8x-5y=14$, $xy=80$.
6. $x^2+3xy+y^2=11$, $x+3y=7$.
7. $2x^2+3y^2-xy=31$, $x-y=3$.
8. $4x^2-xy=90$, $2x+3y=16$.
9. $x^2+y^2=2740$, $x+y=74$.
10. $x^2+y^2=1066$, $x-y=4$.
11. $x+y=7xy$, $12xy=1$.
12. $4x^2+9y^2=100$, $2x-3y=2$.
13. $x^2-y^2=102$, $x-y=3$.
14. $4x^2+121y^2=137$, $2x+11y=15$.
15. $16x^2=14xy+15y^2+166x+108y+21$, $2x=3y+21$.
16. $4x-9y=1$, $xy=21$.
17. $\frac{x}{y}-\frac{y}{x}=\frac{15}{4}$, $x-y=\frac{3}{2}$.

18. $x+y=a,\ xy=b.$

19. $x-y=a,\ xy=b.$

20. $x^2+y^2=a^2,\ x+y=b.$

21. $x^2+y^2=a^2,\ x-y=b.$

22. $ax+by=p,\ xy=q.$

23. $ax-by=p,\ xy=q.$

SIMULTANEOUS EQUATIONS, SOME OF WHICH ARE OF THE SECOND DEGREE IN THREE UNKNOWNS.

264. The following examples will illustrate the methods of proceeding to solve equations of this class.

1. Solve $x+2y+3z=14$............(1),

$$2xy+3xz=13\text{............(2)},$$

$$yz=6\text{............(3)}.$$

Here (2) may be written

$$(2y+3z)x=13.$$

If now we write u for $2y+3z$, (1) and (2) may be written

$$x+u=14,$$

$$xu=13;$$

from which we find $x=1$, $u=13$; or $x=13$, $u=1$.

Choosing the former solution we have from (2) and (3),

$$2y+3z=13\text{...............(4)},$$

$$yz=6\ \text{...............(5)}.$$

Square (4), subtract 24 times (5), and extract the square root,

$$2y-3z=\pm 5\ \text{............(6)}.$$

Combining (4) and (6) we find $y=2$, $z=3$; or $y=\frac{9}{2}$, $z=\frac{4}{3}$. Thus we have the solutions 1, 2, 3; 1, $\frac{9}{2}$, $\frac{4}{3}$; and by taking $x=13$, $u=1$, we would get two other solutions.

2. Solve

$$x^2+y^2+z^2=61 \ldots\ldots\ldots\ldots(1),$$
$$xy+xz=36 \ldots\ldots\ldots\ldots(2),$$
$$yz=18 \ldots\ldots\ldots\ldots(3).$$

Add the sum of (1) and twice (2) to twice (3); thus,

$$(x+y+z)^2=169$$
$$x+y+z=\pm 13 \ldots\ldots\ldots(4).$$

If now u be written for $y+z$, (4) and (2) may be written

$$x+u=\pm 13,$$
$$xu=36;$$

and the solution is effected, as in Ex. 1.

3. Solve

$$x^2+y^2+z^2=110 \ldots\ldots\ldots\ldots(1),$$
$$x+y+z=18 \ldots\ldots\ldots\ldots(2),$$
$$yz=30 \ldots\ldots\ldots\ldots(3).$$

Add (1) to twice (3);

$$x^2+(y+z)^2=170 \ldots\ldots\ldots(4).$$

If now we write u for $y+z$, (4) and (2) may be written

$$x^2+u^2=170 \ldots\ldots\ldots\ldots\ldots(5),$$
$$x+u=18 \ldots\ldots\ldots\ldots\ldots(6).$$

From the square of (6) subtract (5),

$$2xu=154;$$

and the solution follows as before.

4. Solve $yz=a^2$ ....(1),

$$zx=b^2 \text{ ....(2)},$$

$$xy=c^2 \text{ ....(3)}.$$

Multiply together (1), (2), and (3); thus,

$$x^2y^2z^2= \quad a^2b^2c^2,$$

$$x\,y\,z=\pm a\,b\,c \quad \text{.........(4)}.$$

Dividing (4) by (1), (2), (3), we obtain the two solutions

$$\frac{bc}{a},\ \frac{ca}{b},\ \frac{ab}{c};\ -\frac{bc}{a},\ -\frac{ca}{b},\ -\frac{ab}{c}.$$

EXERCISES, LXVI.

1. $x^2+y^2+z^2=14,\ x+y+z=6,\ yz=6.$
2. $x^2+y^2+z^2=36\frac{1}{2},\ x+y+z=10,\ xy+xz=16.$
3. $xy=12,\ yz=20,\ zx=15.$
4. $x^2+y^2+z^2=21,\ yz+zx+xy=-6,\ x+y-z=-5.$
5. $x^2+y^2+z^2=50,\ x+y+z=12,\ xy+xz=27.$
6. $x^2+y^2+z^2=90,\ x+y+z=6,\ xz=28.$
7. $x^2+y^2+z^2=38,\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{11}{xyz}=0,\ x+z=y-10.$
8. $2yz=a(y+z),\ 2zx=b(z+x),\ 2xy=c(x+y).$
9. $x(y+z)=2a,\ y(z+x)=2b,\ z(x+y)=2c.$
10. $x=ayz,\ y=bzx,\ z=cxy.$

265. Equations which do not come under the head of those already discussed can sometimes be solved by previous methods, if special artifices are adopted to transform the equations into simpler forms. Some of these artifices are exhibited in the following ex-

amples. The successful application, however, of such artifices can only be acquired by long experience; and the student is recommended not to spend too much time at such exercises.

1. Solve $$5x^2+3xy=26\dots\dots\dots(1),$$
$$3y^2+2xy=\ 7\dots\dots\dots(2).$$

Whenever, as in this case, all the terms involving the unknowns are of the second degree, assume $y=vx$; (1) and (2) then become

$$x^2\,(5\ \ +3v)=26\dots\dots\dots(3),$$
$$x^2\,(3v^2+2v)=\ 7\dots\dots\dots(4);$$

dividing (3) by (4), and clearing of fractions, we obtain

$$78v^2+31v-35=0,$$

the roots of which are $\frac{1}{2}$ and $-\frac{35}{39}$.

Substituting these values successively in (3), or (4), we get $x=\pm 2$, and therefore $y=\pm 1$;

$$\text{or } x=\pm\frac{13}{15}\sqrt{15}, \text{ and therefore } y=\pm\frac{7}{9}\sqrt{15}.$$

2. Solve $$13y\sqrt{\frac{x^2}{y}+3}=6x^2+20y\dots\dots\dots(1),$$
$$16x^2+y^2=10xy\dots\dots\dots\dots(2).$$

Here (2) is homogeneous and of the second order. In such a case the ratio of x to y can always be determined. Thus let $y=rx$, and substitute in (2).

$$r^2-10r+16=0,$$

$$r=8, \text{ or } 2.$$

Therefore $$y=8x, \text{ or } y=2x.$$

Substitute either of these values in (1) and a quadratic in x results.

3. Solve $$x^3+y^3=133 \text{.........}(1),$$

$$x+y=\quad 7 \text{.........}(2).$$

Assume $x=u+v$, $y=u-v$; (1) and (2) then become

$$2u^3+6uv^2=133 \text{.........}(3),$$

$$2u=7 \text{.........}(4).$$

Substitute for u in (3) its value $\frac{7}{2}$, and we obtain $v=\pm\frac{3}{2}$.

Whence the values of x and y may be found.

4. Solve $$x^3+y^3=18xy \text{............}(1),$$

$$x+y=12 \quad \text{............}(2).$$

Assume $x+y=u$, $xy=v$; (1) and (2) may then be written $$u^3-3uv=18v \text{............}(3),$$

$$u=12 \quad \text{............}(4).$$

Substituting for u in (3), we find

$$v=32.$$

Hence $$x+y=12\ldots\ldots\ldots\ldots(5),$$

$$xy=32\ldots\ldots\ldots\ldots(6),$$

the solutions of which can be effected by previous methods.

EXERCISES, LXVII.

1. $x^2+xy=12$, $xy-2y^2=1$.

2. $5x^2+6y^2-12xy=2$, $4x^2+5y^2-10xy=1$.

3. $x^3+y^3=225y$, $x^2-y^2=75$.

4. $x^3+y^3=189$, $x+y=9$.

5. $x^3-y^2=386$, $x-y=2$.

6. $4y^2=3x^2+xy$, $x^2+y^2=25$.

7. $\frac{x}{y}+\frac{y}{x}=\frac{29}{10}$, $xy=10$.

8. $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$, $xy=6$.

9. $\frac{2}{x}-\frac{3}{y}=5$, $xy=1$.

10. $\frac{9x}{y}=\frac{4y}{x}$, $3xy+2x+y=485$.

11. $xy-2x+3y=21$, $x^2-7xy+12y^2=0$.

12. $x^2-2xy-y^2=31$, $x^2+4xy-2y^2=202$.

PROBLEMS.

266. In the following examples and exercises the statement of the conditions of the problem will lead to simultaneous equations, one or both of which may be of a higher order than the first.

1. A number consists of two digits whose sum is 10, and such that, if it be multiplied by the difference of its digits, the product will be 128. Find the number.

Let x denote the digit in the ten's place, and y the digit in the unit's place. Then the number is $10x+y$.

Hence by the conditions of the problem,

$$x+y=10 \ldots\ldots\ldots\ldots (1),$$

$$(10x+y)(x-y)=128 \ldots\ldots (2);$$

whence we obtain $x=6,\ y=4$.

Hence the required number is 64

2. Two porters A and B drink from a cask of beer for 2 hours, after which A falls asleep and B drinks the remainder in 2 hours and 48 minutes; but if B had fallen asleep and A had continued to drink, it would have taken him 4 hours and 40 minutes to finish the cask. In what time would they be able to drink it separately?

Let x and y denote the numbers of hours it would take A and B respectively to finish the cask, containing g gallons; then $\frac{g}{x}, \frac{g}{y}$ represent the quantities which they respectively drink in one hour. Hence the quantity remaining after two hours' joint drinking is $g\left(1-\frac{2}{x}-\frac{2}{y}\right)$; and if this be divided by $\frac{g}{y}$ it will give the number of hours in which B would drink it alone. Therefore by the conditions of the problem

$$\left(1-\frac{2}{x}-\frac{2}{y}\right)y=2\tfrac{4}{5}\text{.........}(1),$$

$$\left(1-\frac{2}{x}-\frac{2}{y}\right)x=4\tfrac{2}{3}\text{.........}(2).$$

These equations are equivalent to

$$5xy-10y=24x\text{.........}(3),$$

$$3xy-\ \ 6x=20y\text{.........}(4),$$

from which we find $x=10$, $y=6$.

3. Bought different kinds of cloth for £6 10s. at 10s. and 12s. per yard; but if the cost per yard of each kind of cloth had been exactly equal to the number of yards purchased, the cost would have amounted to £3 14s. only. How many yards of each kind were purchased?

Let x and y represent the numbers of yards purchased at 12s. and 10s. respectively; then by the conditions of the problem

$$12x+10y=130\text{.........}(1),$$

$$x^2+\ \ y^2=\ \ 74\text{.........}(2).$$

Solving these equations we find $x=5$, $y=7$; or, $x=\frac{475}{61}$, $y=\frac{223}{61}$.

Exercises, LXVIII.

1. There is a number consisting of two digits, whose product is equal to twice their sum; but if the digits be inverted, the number will exceed the sum of its digits by 54. What is the number?

2. Find two numbers such that their sum shall be 60, and the sum of their squares 1808.

3. Find two numbers whose product shall be 272, and the sum of squares 545.

4. Find two numbers whose product shall be $26\frac{1}{4}$, and the difference of whose squares shall be 44.

5. The difference of the sides of a rectangular field, whose area is one acre, is found to be 33 yards. Calculate the sides.

6. Divide the number 29 into two parts, such that the sum of their cubes shall be 7859.

7. Find two consecutive numbers such that the cube of the greater shall exceed the cube of the less by 271.

8. Two cubical vessels together hold 407 cubic inches; when one vessel is placed on the other, the total height is 11 inches. Find the contents of each.

9. Find two numbers such that their sum, product, and difference of squares shall be equal to each other.

10. Find two numbers such that their sum, product, and the sum of their squares shall be equal to each other.

11. There are two numbers whose product is equal to the difference of their squares, and the sum of whose squares is equal to the difference of their cubes. Find the numbers.

12. There are two numbers the sum of whose squares is 170, and such that their product exceeds their sum by 59. Find the numbers.

13. A number consists of two digits, the difference of whose squares is 40; and, if it be multiplied by the number consisting of the same digits taken in reversed order, the product will be 2701. Find the number.

14. A number consists of two digits whose product is 6, and such that the difference between the square of the number and the square of the number consisting of the same digits in reversed order is equal to 495. Find the number.

15. Find a fraction, the sum of whose numerator and denominator is 11, and such that if it be taken from another fraction whose numerator and denominator are each greater by 2, the difference will be $\frac{1}{24}$.

16. The sum of the squares of the numerator and denominator of a fraction is 389, and the difference of the fraction and its reciprocal is $\frac{189}{170}$. Find the fraction.

17. There is a number such that if 63 be subtracted from it, its digits will be inverted; and if it be multiplied by the sum of its digits, the product will be 1012. Find the number.

18. A number consists of two digits; if these be reversed in order, and the two numbers multiplied

together, the product is 8722; and if the first number be divided by the second, the quotient is 1, and the remainder consists of one figure. What is the number?

19. Find two numbers such that their sum multiplied by the sum of their squares shall be 272, while their difference multiplied by the difference of their squares shall be 32.

20. A person has £13,000, which he divides into two parts, and placing each at interest receives an equal income. If he placed the first sum at the rate of interest of the second, he would receive £360 income; and if he placed the second sum at the rate of the first, he would receive £490 income. What are the two sums, and what the rates of interest?

21. There is a fraction such that if the numerator be increased and the denominator diminished by 2, the reciprocal of the fraction will be the result; while if the numerator be diminished, and the denominator increased by 2, a result will be obtained less than the reciprocal by $\frac{16}{15}$. What is the fraction?

22. If £300 be laid out at simple interest for a certain number of years, it will amount to £360. If the same be allowed to remain two years longer, and at a rate of interest one per cent. higher, it will amount to £405. Find the rate of interest, and number of years for the first sum.

23. A cistern is half full; by one cock it can be filled in a certain time, and by another emptied in a certain time. If both cocks are opened, it will be emptied in 12 hours; but if both cocks are partially closed, so that each would take half-an-hour more to fill or empty the half-full cistern, then, if both are allowed to run, it will take $15\frac{3}{4}$ hours to empty the

cistern. In what time would the first cock alone fill, and in what time would the second alone empty the full cistern?

INEQUALITIES.

267. From the definition laid down in Art. 45 we infer that one quantity a is greater or less than another b, according as the difference $a-b$ is positive or negative.

Thus, $a^2+b^2>2ab$, because the difference a^2+b^2-2ab is equal to $(a-b)^2$, a positive quantity; and $1+4x<x^2+6$, because the difference $4x-x^2-5$ is equal to $-(x-2)^2-1$, a negative quantity.

268. From given inequalities we can deduce others by transformations conducted according to the laws stated in the following Articles. These laws are immediately deducible from the preceding definition.

269. If $a \gtrless b$, then $ma \gtrless mb$, if m be positive.

Thus, from the inequality, $a>b$ we infer $5a>5b$, and $\frac{5}{6}a>\frac{5}{6}b$.

270. If $a \gtrless b$, then $ma \lessgtr mb$, if m be negative.

Thus if $a<b$, then $-4a>-4b$, and $-\frac{2}{3}a>-\frac{2}{3}b$.

271. If $a \gtrless b$, then $a+c \gtrless b+c$, where c is positive, or negative.

Thus if $a>b$, then $a-b>0$, and conversely.

Hence quantities may be transposed from one side to the other by changing signs, as in equations.

272. If $a>b$, then $\frac{1}{a} \lessgtr \frac{1}{b}$ according as a and b have

like or unlike signs. For the difference $\frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}$, which since $b-a$ is negative, is negative or positive according as ab is positive or negative, that is, according as a and b have like or unlike signs.

273. In like manner it may be shown that if $a<b$, then $\frac{1}{a} \overset{>}{<} \frac{1}{b}$ according as a and b have like or unlike signs.

274. If $a_1>b_1$, $a_2>b_2$, &c., then a_1+a_2+ &c. $>b_1+b_2+$&c.

For the difference $a_1+a_2-\ldots-(b_1+b_2+\ldots)=(a_1-b_1)+(a_2-b_2)+\ldots$, which is positive.

So also, if $a_1<b_1$, $a_2<b_2$, &c., then a_1+a_2+ &c. $<b_1+b_2+$&c.

Thus, since $b^2+c^2>2bc$, $c^2+a^2>2ca$, and $a^2+b^2>2ab$, we have $2(a^2+b^2+c^2)>2(bc+ca+ab)$; and therefore

$$a^2+b^2+c^2>bc+ca+ab.$$

275. If $a \overset{>}{<} b$, then $a^{2n+1} \overset{>}{<} b^{2n+1}$, where n is a positive whole number; for the members of the latter inequality have the same signs as those of the former.

Thus from $a>b$ we infer that $a^3>b^3$, $a^7>b^7$, &c.

We cannot, however, infer that $a^2>b^2$. For example, although $4>-5$, it is not true that $4^2>(-5)^2$; and although $-2>-4$, it is not true that $(-2)^2>(-4)^2$.

276. If $a^2 > b^2$, then a cannot lie between $-b$ and b; and if $a^2 < b^2$, a must lie between $-b$ and $+b$.

Thus, if $a^2 > 9$, a cannot lie between -3 and $+3$; if $a^2 < 16$, a must lie between -4 and $+4$, that is $a > -4$ and $< +4$.

277. By the aid of this proposition, and the condition for the existence of real roots in a quadratic, we can find the limits to the real values of a fraction of the form $\dfrac{ax^2+bx+c}{px^2+qx+r}$, as in the following Examples:

1. Find the greatest and least values of $\dfrac{x^2-4x+3}{x^2-2x+4}$.

Let $\dfrac{x^2-4x+3}{x^2-2x+4} = y$; then

$$(1-y)x^2 - 2(2-y)x + 3 - 4y = 0.$$

Now if x be real, we must have

$$(2-y)^2 \geqq (1-y)(3-4y),$$

$$\text{or, } (2y-1)^2 \leqq \frac{7}{3},$$

$$\text{Hence } \quad 2y-1 \geqq -\sqrt{\frac{7}{3}} \text{ and } \leqq +\sqrt{\frac{7}{3}},$$

$$\text{or, } y \geqq \frac{1}{2}\left(1-\sqrt{\frac{7}{3}}\right) \text{ and } \leqq \frac{1}{2}\left(1+\sqrt{\frac{7}{3}}\right).$$

2. Find a positive number such that the sum of it and its reciprocal shall be a minimum.

Let x be the number; then $x + \dfrac{1}{x} = y$ is to be a minimum.

Therefore, in the equation $x^2 - yx + 1 = 0$, we must have $y^2 \geqq 4$; and, therefore, the minimum value of y is 2, and the value of x corresponding is 1.

3. If a, b, c be positive, prove $\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}>a+b+c$.

Since $b^2+c^2>2bc$, we have in this case, dividing by bc,

$$\frac{b}{c}+\frac{c}{b}>2 \dots\dots\dots\dots\dots\dots (1),$$

similarly,
$$\frac{c}{a}+\frac{a}{c}>2 \dots\dots\dots\dots\dots\dots (2),$$

$$\frac{a}{b}+\frac{b}{a}>2 \dots\dots\dots\dots\dots\dots (3).$$

Now multiply (1) by a, (2) by b, and (3) by c, and we obtain, by addition,

$$2\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)>2(a+b+c);$$

therefore
$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}>a+b+c.$$

Exercises, LXIX.

1. Prove that $4<x^2-8x+20$.

2. Prove that $\frac{a}{b}+\frac{b}{a}>2$, if a and b have like signs.

3. Find the values between which x must not lie in order that $4x^2+4x-1$ may be positive.

4. Divide a line a into two parts, such that the rectangle under the parts shall be a maximum.

5. Divide a line a into two parts, such that the sum of the squares on the parts shall be a minimum.

6. Prove that $\frac{x^2-x+1}{x^2+x+1}$ lies between 3 and $\frac{1}{3}$.

7. Prove that $\frac{x^2-7x+6}{x-10}$ cannot lie between 1 and 25.

8. Prove that $\frac{9x^2+6x-7}{12x-8}$ cannot lie between 1 and 2.

9. Prove that $\frac{a^2}{x}+\frac{b^2}{a-x}$ cannot lie between $\frac{(a-b)^2}{a}$ and $\frac{(a+b)^2}{a}$.

10. Prove that $\frac{2x-7}{2x^2-2x-5}$ can have no value between $\frac{1}{11}$ and 1.

11. Prove that the greatest value which $a \cdot \frac{x+a}{x^2+a^2}$ admits of is $\frac{\sqrt{2}+1}{2}$.

12. If a, b, c be positive, prove
$$(b^3+c^3)\,a^2+(c^3+a^3)\,b^2+(a^3+b^3)\,c^2>2abc\,(a^2+b^2+c^2).$$

RATIO AND PROPORTION.

278. The relation which one quantity a bears to another b, as measured by the fraction $\frac{a}{b}$, is called the *ratio* of these quantities, and is expressed either as a fraction $\frac{a}{b}$, or by the notation $a:b$ (read *a to b*).

Thus the ratio of 3 to 4 is expressed by $\frac{3}{4}$, or $3:4$.

279. The quantities which constitute a ratio are called its terms, the first term being called the *antecedent*, and the second the *consequent*.

Thus in the ratio $a:b$, or $\frac{a}{b}$, the first term a is the antecedent, and the second term b is the consequent.

280. The ratio $a : b$ is said to be greater than, equal to, or less than $c : d$, according as $\frac{a}{b} \overset{>}{\underset{<}{=}} \frac{c}{d}$.

281. Ratios are, therefore, compared by reducing the fractions which express them to a common denominator, and comparing the numerators.

282. A ratio is called a ratio of *greater inequality*, of *less inequality*, or of *equality*, according as the antecedent is greater than, less than, or equal to, the consequent.

Thus the ratio $\frac{4}{3}$ is a ratio of greater inequality; $\frac{2}{5}$ is a ratio of less inequality; and $\frac{3}{3}$ is a ratio of equality.

283. A ratio of greater inequality is diminished or increased, and of less inequality increased or diminished, according as each term of the ratio is increased or diminished by the same quantity.

Let $\frac{a}{b}$ be the given ratio. If x be added to each term we get the ratio $\frac{a+x}{b+x}$, which is $\overset{<}{>} \frac{a}{b}$ according as

$$ab+bx \overset{<}{>} ab+ax,$$

that is, as

$$bx \overset{<}{>} ax.$$

Now (1) if $a>b$, then $bx<ax$, if x is positive; and $bx>ax$, if x is negative. In the former case the ratio is diminished, and in the latter increased. (2) If $a<b$, $bx<ax$, if x is negative, and $bx>ax$, if x is positive. In the former case the ratio is diminished, and in the latter increased.

Thus if 2 be added to each term of $\frac{4}{3}$, we get $\frac{6}{5}$, and the ratio is diminished; and if 1 be taken from each term we get $\frac{3}{2}$, and the ratio is increased.

284. The ratio $\frac{ac}{bd}$ is said to be *compounded* of the ratios $\frac{a}{b}, \frac{c}{d}$; the ratio $\frac{ace}{bdf}$ is said to be *compounded* of the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$; and so, generally, any ratio is said to be compounded of the ratios expressed by the fractions whose product expresses the given ratio.

Thus the ratio compounded of $\frac{2}{3}$ and $\frac{4}{5}$ is $\frac{8}{15}$.

285. The ratios $\frac{a^2}{b^2}, \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}; \frac{a^3}{b^3}, \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$, &c., are called the *duplicate, sub-duplicate; triplicate, sub-triplicate*, &c., ratios, respectively, of $\frac{a}{b}$.

Thus the duplicate ratio of $\frac{2}{3}$ is $\frac{4}{9}$; and the sub-triplicate ratio of $\frac{8}{27}$ is $\frac{2}{3}$.

286. When two ratios are equal they constitute a *proportion;* and the four terms are said to be *proportionals.*

Thus if $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$, the four quantities a, b, c, d are said to be proportionals, the equality between the ratios being expressed by ::, or =.

287. In this case a and d are called the extremes, b and c the *means.*

288. From the proportion $\frac{a}{b}=\frac{c}{d}$, we can deduce certain other proportions, as follows:

(i.) Since $ad=bc$, we obtain on dividing by cd,

$$\frac{a}{c}=\frac{b}{d}\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

(ii.) Since $bc=ad$, we obtain on dividing by ab,

$$\frac{c}{a}=\frac{d}{b}\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

(iii.) By adding 1 to each ratio of the given proportion we get

$$\frac{a}{b}+1=\frac{c}{d}+1,$$

$$\text{or, } \frac{a+b}{b}=\frac{c+d}{d}\ \ldots\ldots\ldots\ldots(3).$$

(iv.) By subtracting 1 from each ratio of the proportion we get

$$\frac{a}{b}-1=\frac{c}{d}-1,$$

$$\text{or, } \frac{a-b}{b}=\frac{c-d}{d}\ \ldots\ldots\ldots\ldots(4).$$

(v.) By dividing (3) by (4) we obtain

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}\ \ldots\ldots\ldots\ldots(5).$$

In Geometrical language (1), (2), (3), (4), (5) are said to be proportions which follow from the given proportion by taking its constituents a, b, c, d *alternando, invertendo, componendo, dividendo, componendo et dividendo*, respectively.

289. If $\frac{a}{b}=\frac{c}{d}$, and $f(a, b)$, $\phi(a, b)$ denote any two homogeneous functions of n dimensions in a, b, then

$$\frac{f(a, b)}{\phi(a, b)}=\frac{f(c, d)}{\phi(c, d)}.$$

For let $\frac{a}{b}=\frac{c}{d}=x$; then $a=bx$, $c=dx$.

Substituting these values for a and c in $f(a, b)$, &c., we obtain $f(a, b)=f(bx, b)=b^n f(x, 1)$, since every term of $f(bx, b)$ is of n dimensions in b. In like manner we get $\phi(a, b)=b^n\phi(x, 1)$; $f(c, d)=d^n f(x, 1)$; $\phi(c, d)=d^n\phi(c, 1)$.

Hence $$\frac{f(a, b)}{\phi(a, b)}=\frac{b^n f(x, 1)}{b^n\phi(x, 1)}=\frac{f(x, 1)}{\phi(x, 1)};$$

and $$\frac{f(c, d)}{\phi(c, d)}=\frac{d^n f(x, 1)}{d^n\phi(x, 1)}=\frac{\phi(x, 1)}{f(x, 1)}.$$

Therefore, $$\frac{f(a, b)}{\phi(a, b)}=\frac{f(c, d)}{\phi(c, d)}.$$

EXAMPLES.

1. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^2+ab-b^2}{2ab-3b^2}=\frac{c^2+cd-d^2}{2cd-3d^2}$.

Here the terms of the last proportion are homogeneous functions of two dimensions.

Let $\frac{a}{b}=\frac{c}{d}=x$, and substitute bx for a, and dx for c.

Then $$\frac{a^2+ab-b^2}{2ab-3b^2}=\frac{b^2x^2+b^2x-b^2}{2b^2x-3b^2}=\frac{x^2+x-1}{2x-3};$$

and $$\frac{c^2+cd-d^2}{2cd-3d^2}=\frac{d^2x^2+d^2x-d^2}{2d^2x-3d^2}=\frac{x^2+x-1}{2x-3}.$$

Therefore $$\frac{a^2+ab-b^2}{2ab-3b^2}=\frac{c^2+cd-d^2}{2cd-3d^2}.$$

2. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a^2+3ab+b^2}{c^2+3cd+d^2}=\frac{2ab+3b^2}{2cd+3d^2}$.

As in Ex. 1 prove

$$\frac{a^2+3ab+b^2}{2ab+3b^2}=\frac{c^2+3cd+d^2}{2cd+3d^2};$$

and, therefore, by (i.) Art. 288.

$$\frac{a^2+3ab+b^2}{c^2+3cd+d^2}=\frac{2ab+3b^2}{2cd+3d^2}.$$

Exercises, LXX.

1. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{2a-3b}{5a+4b}=\frac{2c-3d}{5c+4d}$.

2. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{5a^3-3ab^2}{2a^2b+7b^3}=\frac{5c^3-3cd^2}{2c^2d+7d^3}$.

3. If $\frac{x}{y}=\frac{z}{u}$, prove $\frac{2x^2-7xy}{2z^2-7zu}=\frac{3xy+8y^2}{3zu+8u^2}$.

4. If $\frac{x}{y}=\frac{a^2}{b^2}$, prove $\frac{5a^4+7b^4}{5x^2+7y^2}=\frac{7a^4-5b^4}{7x^2-5y^2}$.

290. If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\dots=\frac{a_n}{b_n}$, then each of these ratios is equal to

$$\left\{\frac{m_1a_1^n+m_2a_2^n+m_3a_3^n+\dots+m_na_n^n}{m_1b_1^n+m_2b_2^n+m_3b_3^n+\dots+m_nb_n^n}\right\}^{\frac{1}{n}},$$

where m_1, m_2, m_3, &c., are any positive or negative quantities.

For let $\frac{a_1}{b_1}=\frac{a_2}{b_2}=$&c.,$=x$.

Then $a_1=b_1x$, $a_2=b_2x$, &c.

Therefore $a_1^n=b_1^nx^n$, $a_2^n=b_2^nx^n$, &c.;

and $m_1a^n=m_1b_1^nx^n$, $m_2a_2^n=m_2b_2^nx^n$, &c.

Adding the last set of equalities, we obtain

$$m_1a_1^n+m_2a_2^n+\ldots=(m_1b_1^n+m_2b_2^n+\ldots)x^n;$$

therefore $$\frac{m_1a_1^n+m_2a_2^n+\ldots}{m_1b_1^n+m_2b_2^n+\ldots}=x^n;$$

and, therefore, extracting the *n*th root, we get

$$\left\{\frac{m_1a_1^n+m_2a_2^n+\ldots\ldots}{m_1b_1^n+m_2b_2^n+\ldots\ldots}\right\}^{\frac{1}{n}}=x=\frac{a_1}{b_1}=\frac{a_2}{a_2}=\&c.$$

EXAMPLES.

1. If $\frac{a}{b}=\frac{c}{d}$, each of these ratios is equal to $\left(\frac{a^2+c^2}{b^2+d^2}\right)^{\frac{1}{2}}$.

Here $m_1=m_2=1$, and $n=2$.

2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, each of these ratios is equal to

$$\frac{a^2-c^2+e^2}{ab-cd+ef}.$$

Here $m_1=a$, $m_2=-c$, $m_3=e$, and $n=1$.

3. If $\frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}$, each of these ratios is equal to $\frac{x+y+z}{2x+2y+2z}=\frac{1}{2}$.

Here $m_1=m_2=m_3=1$, and $n=1$.

4. If $\frac{a}{b+c}=\frac{b}{c+a}=\frac{c}{a+b}$, prove $a=b=c$.

We have, as in Ex. 3, each ratio$=\frac{1}{2}$; so that

$$2a=b+c\ldots\ldots\ldots\ldots(1),$$

$$2b=c+a\ldots\ldots\ldots\ldots(2),$$

$$2c=a+b\ldots\ldots\ldots\ldots(3).$$

Subtracting (2) from (1) we get $2a-2b=b-a$, and therefore $a=b$. Similarly $a=c$.

Hence $a=b=c$.

5. If $\frac{b^2+c^2-a^2}{bc}=\frac{c^2+a^2-b^2}{ca}=\frac{a^2+b^2-c^2}{ab}$, prove each ratio equal to 1.

By taking the first and second ratios together, we have each ratio equal to

$$\frac{b^2+c^2-a^2+c^2+a^2-b^2}{bc+ca}=\frac{2c^2}{bc+ca}=\frac{2c}{a+b}.$$

Similarly by taking the second and third, and third and first ratios together, we get each ratio equal to

$$\frac{2a}{b+c}, \text{ or } \frac{2b}{c+a}.$$

Hence each of the given ratios is equal to

$$\frac{2a}{b+c}=\frac{2b}{c+a}=\frac{2c}{a+b},$$

each of which is equal to

$$\frac{2a+2b+2c}{2a+2b2c}, \text{ or } 1.$$

EXERCISES, LXXI.

1. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{(a+c)\,(a^2+c^2)}{(a-c)\,(a^2-c^2)}=\frac{(b+d)\,(b^2+d^2)}{(b-d)\,(b^2-d^2)}$.

2. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove $(b+c)\,(b+d)=(c+a)\,(c+d)$.

3. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=$&c., prove that each of these ratios is equal to

$$\frac{a\,(a-b)+b\,(b-c)+c\,(c-d)+\&c.}{a\,(b-c)+b\,(c-d)+c\,(d-e)+\&c.}$$

4. If the ratio of $a+x:a-x$ equals the duplicate ratio of $a+b:a-b$, then

$$\frac{x}{a}=\frac{2ab}{a^2+b^2}.$$

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that $\frac{ac+ce+ea}{bd+df+fb}=\frac{a^2}{b^2}$,

and $\frac{\sqrt{ma}+\sqrt{nb}}{\sqrt{ma}-\sqrt{nb}}=\frac{\sqrt{mc}+\sqrt{nd}}{\sqrt{mc}-\sqrt{nd}}=\frac{\sqrt{me}+\sqrt{nf}}{\sqrt{me}-\sqrt{nf}}$.

6. If $\frac{a}{b}=\frac{c}{d}$, shew that $\frac{\sqrt{a-b}}{\sqrt{c-d}}=\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}-\sqrt{d}}$.

7. If $\frac{a_1}{a_2}=\frac{a_2}{a_3}=\frac{a_3}{a_4}=\ldots=\frac{a_n}{a_{n+1}}$, shew that

$$\left(\frac{a_1+a_2+a_3+\ldots+a_n}{a_2+a_3+a_4+\ldots+a_{n+1}}\right)^n=\frac{a_1}{a_{n+1}}.$$

8. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a^2+c^2}{b^2+d^2}=\frac{\sqrt{a^4+c^4}}{\sqrt{b^4+d^4}}$.

9. If $\frac{2y-z}{2b+c}=\frac{2z-x}{2c+a}=\frac{2x-y}{2a+b}$, shew that

$$21\,(a+b+c)\,(x+2y+3z)=(41a+38b+47c)\,(x+y+z).$$

10. If $\frac{x+y}{3a-b}=\frac{y+z}{3b-c}=\frac{z+x}{3c-a}$, prove that

$$\frac{x+y+z}{ax+by+cz}=\frac{a+b+c}{a^2+b^2+c^2}.$$

11. If $\frac{a+c}{b}=\frac{c}{a}=\frac{a}{c-b}$, determine the ratios $a:b:c$.

12. If $\frac{b}{a+b}=\frac{a+c-b}{b+c-a}=\frac{a+b+c}{2a+b+2c}$, determine the ratios $a:b:c$.

13. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, prove that

$$(a+b+c)\,(yz+zx+xy)=(x+y+z)\,(ax+by+cz).$$

14. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that

$$(ac+ce+ea)^2\,(b^2+d^2+f^2)=(bc+de+fa)^2\,(a^2+c^2+e^2).$$

15. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\dots$, shew that

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}+\ldots=$$

$$\frac{(x+y+z+\ldots)^2+(a+b+c+\ldots)^2}{x+y+z+\ldots\ +\ a+b+c+\ldots}.$$

16. If $\frac{ny+mz}{a}=\frac{lz+nx}{b}=\frac{mx+ly}{c}$, prove that

$$\frac{l}{x\ (by+cz-ax)}=\frac{m}{y\ (cz+ax-by)}=\frac{n}{z\ (ax+by-cz)}.$$

291. If $\frac{a}{b}=x$, a finite quantity, and $a=0$, then $b=0$; and conversely.

For $bx=a=0$, and therefore $b=0$, since x does not vanish.

Hence, if $\frac{a_1}{b_2}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\ldots=\frac{a_n}{b_n}=x$, a finite quantity,

and $m_1a_1^n+m_2a_2^n+\ldots+m_na^n=0$, then

$$m_1b_1^n+m_2b_2^n+\ldots+m_nb_n^n=0.$$

For each of the given ratios is equal to

$$\left\{\frac{m_1a_1^n+m_2a_2^n+\ldots}{m_1b_1^n+m_2b_2^n+\ldots}\right\}^{\frac{1}{n}},$$

the numerator of which vanishes by the given condition; therefore the denominator also vanishes, and hence

$$m_1b_1^n+m_2b_2^n+\ldots=0.$$

EXAMPLES.

1. If $\frac{a}{b-c}=\frac{b}{c-a}=\frac{c}{a-b}$, then $a+b+c=0$, because the sum of $b-c$, $c-a$, $a-b$ vanishes.

2. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, prove $(b-c)\,x+(c-a)\,y+(a-b)\,z=0$.

Here each ratio is equal to

$$\frac{(b-c)\,x+(c-a)\,y+(a-b)\,z}{(b-c)\,(b+c-a)+(c-a)\,(c+a-b)+(a-b)\,(a+b-c)},$$

the denominator of which vanishes identically; therefore the numerator also vanishes.

3. If $\frac{a^2}{x^2-yz}=\frac{b^2}{y^2-zx}=\frac{c^2}{z^2-xy}$, prove that

$$a^2x+b^2y+c^2z=(a^2+b^2+c^2)\,(x+y+z).$$

In this case each of the given ratios is equal to

$$\frac{a^2x+b^2y+c^2z}{x\,(x^2-yz)+y\,(y^2-zx)+z\,(z^2-xy)}\text{.........}(1),$$

and also to $\frac{a^2+b^2+c^2}{x^2+y^2+z^2-yz-zx-xy}$,

or $$\frac{(a^2+b^2+c^2)\,(x+y+z)}{(x^2+y^2+z^2-yz-zx-xy)\,(x+y+z)}\text{.........}(2).$$

Therefore, combining (1) and (2), each of the given ratios is equal to

$$\frac{a^2x+b^2y+c^2z-(a^2+b^2+c^2)(x+y+z)}{x^3+y^3+z^3-3xyz-(x^2+y^2+z^2-yz-zx-xy)(x+y+z)},$$

the denominator of which vanishes identically; therefore the numerator also vanishes.

Exercises, LXXII.

1. If $\frac{bz-cy}{a}=\frac{cx-az}{b}=\frac{ay-bx}{c}$, prove that $ax+by+cz=0$.

2. If $\frac{a+b}{3(a-b)}=\frac{b+c}{4(b-c)}=\frac{c+a}{5(c-a)}$, prove that $32a+35b+27c=0$.

3. If $\frac{a+b}{a-b}=\frac{b+c}{2(b-c)}=\frac{c+a}{3(c-a)}$, prove that $8a+9b+5c=0$.

4. If $\frac{x}{a(y-z)}=\frac{y}{b(z-x)}=\frac{z}{c(x-y)}$, prove that $\frac{x}{a}+y$ $+\frac{z}{c}=0$.

5. If $\frac{x}{a(y+z)}=\frac{y}{b(z+x)}=\frac{z}{c(x+y)}$, prove that

$$\frac{x}{a}(y-z)+\frac{y}{b}(z-x)+\frac{z}{c}(x-y)=0.$$

6. If $\frac{x}{l(mb+nc-la)}=\frac{y}{m(nc+la-mb)}$

$=\dfrac{z}{n\,(la+mb-nc)}$, prove that

$$\frac{x}{l}(mb-nc)+\frac{y}{m}(nc-la)+\frac{z}{n}(la-mb)=0.$$

7. If $\dfrac{a}{lx\,(ny-mz)}=\dfrac{b}{my\,(lz-nx)}=\dfrac{c}{nz\,(mx-ly)}$, prove that

$$\frac{a}{lx}(l-x)+\frac{b}{my}(m-y)+\frac{c}{nz}(n-z)=0.$$

VARIATION.

292. One quantity is said to vary *directly* as another when it bears a constant ratio to that other.

Thus x varies directly as y if $x:y$ is constant, that is, if $x=ky$, where k is a constant quantity.

This is also expressed by the sign of variation, $\propto$, written between the quantities, as $x \propto y$.

Ex. The area A of a triangle is one-half the product of the base b into the height h, or $A=\frac{1}{2}bh$; therefore, if b be kept constant, and h varied, $A \propto h$. So that, if h be doubled, A will be doubled also.

293. One quantity is said to vary *inversely* as another when it bears a constant ratio to the reciprocal of that other.

Thus x varies inversely as y if $x:\dfrac{1}{y}$ is constant, or $x=\dfrac{k}{y}$.

This is also expressed thus: $x \propto \frac{1}{y}$ (read *x varies inversly as y*).

Ex. If the area A of a triangle be kept constant, bh will be constant, and therefore b will vary inversely as h; so that, if b be doubled, for instance, h will be reduced one-half.

294. One quantity is said to vary *jointly* as two others when it varies directly as their product.

Thus, if x varies jointly as y and z, $x \propto yz$, or $x = kyz$, k being constant.

295. If a quantity x varies directly as one y, and inversely as another z, the variation is expressed thus:

$$x \propto \frac{y}{z}, \text{ or } x = \frac{ky}{z},$$

where k is constant.

296. When one quantity varies as two others in such a manner that when either is kept constant it varies as the other; then when both vary it will vary as the two jointly.

Let x vary as y when z is constant, and as z when y is constant.

(1). z being constant, let a, b be the corresponding values of x and y; therefore

$$\frac{x}{a} = \frac{y}{b} \text{..................}(1).$$

(2). y being constant and equal to b, let c, d be corresponding values of x and z; therefore

$$\frac{c}{a} = \frac{d}{z} \text{..................}(2).$$

Hence, dividing (1) by (2) we get

$$\frac{x}{c}=\frac{yz}{bd}\text{, or } x=\frac{c}{bd}\cdot yz\text{;}$$

and therefore $x \propto yz$.

As an example, we may take the area of a triangle ABC, of which BC is the base, and A the vertex.

Let $BC=b$, and let the distance of A from BC be h.

First, let BC be changed to $B'C'=b'$, h being kept constant; therefore

$$\frac{AB'C'}{ABC}=\frac{b'}{b}\text{..................(1).}$$

Secondly, let A be then changed to A′, B′C′ being kept constant; therefore

$$\frac{A'B'C'}{AB'C'}=\frac{h'}{h}\text{..................(2),}$$

h' being the distance of A′ from B′C′.

Multiply (1) by (2), and we obtain

$$\frac{A'B'C'}{ABC}=\frac{b'h'}{bh}\text{;}$$

$$\text{or, } A'B'C'=\frac{ABC}{bh}\cdot b'h'\text{;}$$

and therefore the area varies jointly as the base and height.

EXAMPLES.

1. If $x \propto y$ and $y \propto z$, then $x \propto z$.

For let $x=my$, and $y=nz$, m and n being constant;

$$\text{therefore } x=mnz\text{;}$$

$$\text{or } x \propto z,$$

since mn is constant.

2. If $x \propto y$ and $x \propto z$, then $x \propto \sqrt{yz}$.

For let $x=my$, $x=nz$; therefore

$$x^2=mnyz,$$

$$\text{or } x=\sqrt{mn}\,.\,\sqrt{yz}.$$

Therefore $x \propto \sqrt{yz}$, since $\sqrt{mn}$ is constant.

3. A varies as B and C jointly; and A=1 when B=1 and C=1; find the value of A when B=2 and C=2.

Here $A=mBC$, where m is some constant.

To determine m write for A, B, C their corresponding values 1, 1, 1; therefore $m=1$, and $A=BC$.

Hence if $B=2$, and $C=2$, $A=4$.

4. If £70 pay 10 men for 35 days' work, for how many days will £120 pay 30 men?

Let x, y, z denote the wages, number of men, and days, respectively. Then since the amount of wages varies jointly as the number of men and days, we have

$$x=kyz,$$

where k is some constant.

Substitute for x, y, z their corresponding values 70, 10, 35; therefore

$$70=k\times 10\times 35,$$

$$\text{or, } k=\tfrac{1}{5}.$$

Hence $x=\frac{1}{5}yz$, and therefore if $x=120$, and $y=30$,

$$120=\tfrac{1}{5}\,.\,30z,$$

$$\text{or } z=20.$$

5. The value of diamonds varies as the square of their weight, and the square of the value of rubies varies as the cubes of their weights. A diamond weighing a carats is worth m times a ruby weighing b carats, and both together are worth £p. Find the value of a diamond and of a ruby each in terms of its weight w carats.

Let the value in £ of a diamond of a carats be ha^2, and the value of a ruby of b carats be $kb^{\frac{3}{2}}$, h and k being certain constants to be determined.

By the conditions of the question,

$$a^2h = mkb^{\frac{3}{2}},$$

$$ha^2 + kb^{\frac{3}{2}} = p;$$

two equations in h and k, from which we find

$$h = \frac{mp}{(m+1)a^2}, \quad k = \frac{p}{(m+1)b^{\frac{3}{2}}}.$$

Therefore the value in £ of a diamond of w carats

$$= hw^2 = \frac{mpw^2}{(m+1)a^2};$$

and of a ruby of w carats

$$= kw^{\frac{3}{2}} = \frac{pw^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}.$$

Exercises, LXXIII.

1. If $x \propto y$, and $x=2$ when $y=1$, find the value of x when $y=2$.

2. $3x+5y$ varies as $5x+3y$, and $x=5$ when $y=2$; find the ratio $x : y$.

3. x varies as y and z jointly; and $x=8$ when $y=2$ and $z=2$; find the value of yz when $x=10$.

4. If $y^2 \propto a^2 - x^2$, and $x=0$ when $y=b$, express y in terms of x.

5. Given that $z \propto x+y$, and $y \propto x^2$; and that when $x=1$, $y=2$, and $z=3$; find z in terms of x.

6. The wages of 5 men for 7 weeks being £21 17s. 6d., how many men can be hired to work 4 weeks for £30?

7. If $a+b \propto a-b$, prove that $a^2+b^2 \propto ab$.

8. If the time of describing any space vary as the square root of the space, and if the time of describing 144 feet be 3 seconds, what will be the time of describing 400 feet?

9. Given that y is equal to the sum of two quantities, one of which varies as x, and the other as $\frac{1}{x}$, and that when $x=a$, $y=p$, and when $x=b$, $y=q$; express y in terms of x.

10. If $a \propto b$, show that $(a^2-b^2)(a^3+b^3) \propto (a-b)(a^4+b^4)$.

11. If $a \propto \frac{1}{b}$, show that $a^3b+a^2+\frac{a}{b} \propto \frac{1}{b^2}$.

12. If $x+y+z \propto x+y-z$, and $x^2+y^2+z^2 \propto x^2+y^2-z^2$, show that $x \propto z$ and $y \propto z$.

ARITHMETICAL PROGRESSION.

297. Quantities are said to be in Arithmetical Progression when they increase or decrease by a common difference.

The common difference is the difference between any term after the first and the term immediately preceding.

Thus the series

$$2,\ 5,\ 8,\ 11, \ldots\ldots$$

$$20,\ 18,\ 16,\ 14, \ldots\ldots$$

$$a,\ a+d,\ a+2d,\ a+3d, \ldots\ldots$$

are in Arithmetical Progression, the common differences being 3, -2, and d, respectively.

298. The terms between the first and last are called Arithmetical *means*.

299. If the first term be denoted by a, the common difference by d, the last term by l, and the number of terms by n, then the series of terms will be

$$a,\ a+d,\ a+2d,\ a+3d, \ldots\ldots$$

where it will be observed that the coefficient of d in any term is one less than the number of that term; therefore, for l, the nth term, we have

$$l=a+(n-1)d.$$

Thus the 11th term of $1, 3, 5, \ldots\ldots$ is $1+(11-1)2=21$; the 13th term of $30, 28, 26, \ldots\ldots$ is $30+(13-1)(-2)=6$.

300. To find the sum of a given number of quantities in Arithmetical Progression in terms of (1) a, l, n; (2) a, d, n.

Let s denote the sum; then

$$s=a+(a+d)+(a+2d)+\ldots\ldots+(l-2d)+(l-d)+l.$$

Also $s=l+(l-d)+(l-2d)+\ldots\ldots+(a+2d)+(a+d)+a$, by writing the series in reverse order.

Therefore, by addition, we have

$$2s=(a+l)+(a+l)+\ldots\ldots \text{to } n \text{ terms.}$$

$$=(a+l)n;$$

$$s=(a+l)\frac{n}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Secondly, since $l=a+(n-1)d \ldots (2)$,

we get, on substituting this value in (1),

$$s=\{2a+(n-1)d\}\frac{n}{2} \ldots\ldots (3).$$

The equations (1) and (2) are independent and involve the five quantities a, l, n, d, s; therefore if any three of these quantities be given the remaining two can be found from equations (1) and (2).

EXAMPLES.

1. Find the sum of n terms of the series of odd numbers, 1, 3, 5,……

Here $a=1$, $d=2$; therefore, by formula (3),

$$s=\{2+2(n-1)\}\frac{n}{2}=n^2.$$

2. Insert n Arithmetical means between the given numbers a and b.

Let d denote the common difference.

As there are $n+2$ numbers in the series, the last one b being the $n+2$th, we have, by formula (2),

$$b=a+(n+2-1)d;$$

$$\text{therefore } d=\frac{b-a}{n+1},$$

and hence the means can be found.

If $a=9$, $b=24$, and $n=5$, d will be $2\frac{1}{2}$. Therefore the means in this case are

$$11\tfrac{1}{2},\ 14,\ 16\tfrac{1}{2},\ 19,\ 21\tfrac{1}{2}.$$

3. Find the number of terms in a series of which the first term is 10, the common difference 5, and the sum 175.

Substituting the given values in formulas (1) and (2), we obtain

$$(10+l)n=350,$$

$$l=10+5(n-1).$$

Eliminating l from these equations, we get

$$n^2+3n-70=0,$$

the roots of which are 7 and -10.

From the nature of the case the positive root 7 must be taken.

EXERCISES, LXXIV.

1. Sum the series $2\frac{1}{2}+2\frac{5}{6}+3\frac{1}{6}+\ldots\ldots$to 13 terms.

2. Sum the series $\frac{1}{6}+\frac{1}{3}+\frac{1}{2}+\ldots\ldots$to 8 terms.

3. Sum the series $10+7+4+\ldots\ldots$to 10 terms.

4. Sum the series $2\frac{1}{2}+1\frac{3}{4}+1+\ldots\ldots$to 6 terms.

5. Sum the series $\frac{8}{9}+\frac{29}{63}+\frac{2}{63}+\ldots\ldots$to 6 terms.

6. Sum the series $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\ldots$to n terms.

7. Insert 4 Arithmetic means between $5\frac{1}{2}$ and $6\frac{1}{2}$.

8. The sum of a series in A. P. whose first term is 2 and last term 42 is 198; find the common difference and the number of terms.

9. How many terms of the series $17+15+13+\ldots$ amount to 56?

10. Find what number of terms of the series $6+9+12+\ldots$ will amount to 105.

11. Between the numbers 3 and 27 as extremes, find a series of Arithmetic means whose sum shall be 75.

12. For boring an Artesian well 500 feet deep the cost is 2s. 8d. for the first foot, and a halfpenny in addition for each foot following; what is the cost for boring the last foot, and how much for the whole well?

13. According to the laws of gravity, a heavy body falling from rest descends 16 feet in the first second of time, three times as much in the second, five times as much in the third, seven times as much in the fourth, and so on; through how many feet will it fall in the 13th second, and through how many in the whole 13 seconds?

14. The sum of the squares of the tenth and sixteenth terms of a series in A. P. of which the first term is unity, is equal to $102\frac{1}{2}$; find the terms.

15. The difference of the squares of the sixth and eighteenth terms is 1104, and the first term is unity; find the terms.

16. The sum of the first nine terms of a series in A. P., whose first term is unity, is one-third of the sum of the following nine terms; find the series.

GEOMETRICAL PROGRESSION.

301. Quantities are said to be in Geometrical Progression when they increase or diminish by a common ratio.

302. The common ratio is the ratio of any term after the first to the term immediately preceding it.

Thus the series

$$1, 3, 9, 27, \ldots\ldots\ldots$$

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ \ldots\ldots\ldots$$

$$a, ar, ar^2, ar^3 \ldots\ldots$$

are in Geometrical Progression, the common ratios being 3, $\frac{1}{2}$, and r, respectively.

303. The terms between the first and the last are called Geometrical *means*.

304. If the first term be denoted by a, the common ratio by r, the last term by l, and the number of terms by n, the series of terms will be

$$a, ar, ar^2, ar^3 \ldots\ldots$$

the index of r in any term being one less than the number of the term; therefore, for the last, or nth, term we have

$$l = ar^{n-1} \ldots\ldots\ldots (1).$$

305. To find the sum of a given number of quantities in Geometrical Progression in terms of (1) a, r, n; (2) a, l, r.

Let s denote the required sum; then

$$s = a + ar + ar^2 + \ldots\ldots + ar^{n-1}.$$

Also $$rs = \quad ar + ar^2 + \ldots\ldots + ar^{n-1} + ar^n,$$

by multiplying the first equation by r.

Therefore, by subtraction, we obtain

$$(r-1)\,s = ar^n - a,$$

$$s = a.\frac{r^n - 1}{r-1} \ldots\ldots\ldots (2).$$

Since from (1) we have $lr=ar^n$, we get, on substituting this value in (2),

$$s=\frac{lr-a}{r-1}\text{.........}(3).$$

The equations (1) and (3) are independent, and involve the five quantities a, l, n, r, s; therefore if any three of these quantities be given, the remaining two can be determined from equations (1) and (3).

306. If r be less than unity, r^n decreases as n increases, and becomes indefinitely small as n becomes indefinitely large. This is generally expressed by saying that $r^n=0$ if $n=\infty$ (infinity); by which it is meant that r^n approaches zero as n approaches ∞.

In this case the value of s in (2) continually approximates without limit to the value $\frac{a}{1-r}$ which is called the sum of the infinite series, the value of r^n being neglected.

Hence, r being less than unity, the sum of the nfinite series is given by the formula

$$s=\frac{a}{1-r}\text{.........}(4).$$

Examples.

1. Find the sum of 6 terms of the series 1, 3, 9, 27,...

Here $a=1$, $r=3$, $n=6$; therefore from formula (2) we have $s=\frac{3^6-1}{3-1}=364.$

2. Insert 3 Geometrical means between 2 and 32.

Here the series is to consist of 5 terms, the first of which is 2 and the last 32; therefore if r denote the common ratio, the value of the 5th term is

$$2r^4=32,$$

$$\therefore r=2.$$

Therefore the means are 4, 8, and 16.

3. Find the sum of the series

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots \text{ to infinity.}$$

Here $a=1$, $r=\frac{1}{2}$; therefore, by formula (4),

$$s=\frac{1}{1-\frac{1}{2}}=2;$$

that is, the greater the number of terms in the series, the more nearly does its sum approach 2; and if n be indefinitely large, the difference between the required sum and 2 is less than any assignable quantity.

4. Reduce the circulating decimal ·3242424...to a finite vulgar fraction.

Here $\cdot 32424\ldots=\frac{3}{10}+\frac{24}{10^3}+\frac{24}{10^5}+\ldots$

The terms after the first belong to an infinite series of which the common ratio $=\frac{1}{10^2}$, and sum $=\frac{24}{990}$.

Therefore the value required $=\frac{3}{10}+\frac{24}{990}=\frac{321}{990}$.

Such examples are better solved as follows:

$$\text{Let}\qquad s=\cdot32424\ldots$$

$$\text{therefore}\qquad 10s=3\cdot2424\ldots$$

$$1000s=324\cdot2424\ldots$$

subtracting the last two equalities we get

$$990s=321.$$

$$\text{Therefore}\qquad s=\frac{321}{990}.$$

Exercises, LXXV.

1. Sum the series $4-2+1-\ldots$ to 5 terms and to infinity.

2. Sum the series $\frac{2}{3}-\frac{1}{2}+\frac{3}{8}-\ldots$ to 6 terms and to infinity.

3. Sum the series $9+3+1+\ldots$ to 5 terms and to infinity.

4. Sum the series $\frac{2}{5}-(\frac{2}{5})^{\frac{1}{2}}+1-\ldots$ to 5 terms.

5. Sum the series $(\frac{1}{2})^{-\frac{1}{2}}+1+(\frac{1}{2})^{\frac{1}{2}}+\ldots$ to infinity.

6. Insert three Geometric means between 2 and 162.

7. There are five terms in G. P.; the sum of the

even terms is $4\frac{7}{8}$, and of the odd terms $8\frac{5}{16}$. Find the series.

8. There are seven terms in G. P. such that the sum of the first six terms is $157\frac{1}{2}$, and of the last six 315. Find the numbers.

9. What is the infinite Geometric series of which the sum is $2\frac{1}{12}$ and second term $-\frac{1}{2}$?

10. If a and b are respectively the Arithmetic and Geometric means between two numbers, find the numbers in terms of a and b.

11. Reduce ·81756756... to an equivalent vulgar fraction.

12. Reduce ·3526464... to an equivalent vulgar fraction.

HARMONICAL PROGRESSION.

307. Three quantities a, b, c, are said to be in Harmonical Progression when $a : c :: a-b : b-c$.

308. Any number of quantities are said to be in Harmonical Progression when every three consecutive ones are in H. P

309. The quantities between the first and last terms of a series in H. P. are called Harmonical means.

310. The reciprocals of quantities in H. P. are in A. P.

Let a, b, c be in H. P.; therefore

$$\frac{a}{c}=\frac{a-b}{b-c};$$

or, $$ab-ac=ac-bc;$$

divide by abc,

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a};$$

and therefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

EXAMPLES.

1. Find the Harmonic mean between a and b.

Here we must first insert an Arithmetical mean between $\frac{1}{a}$ and $\frac{1}{b}$. If this be called x, we have

$$2x=\frac{1}{a}+\frac{1}{b},$$

or $$x=\frac{a+b}{2ab}.$$

Hence the required mean $=\frac{1}{x}=\frac{2ab}{a+b}$.

2. If a, b, c are in H. P., prove $\frac{a+c-2b}{a-c}=\frac{a-c}{a+c}$.

Since a, b, c are in H. P., we have

$$\frac{a}{c}=\frac{a-b}{b-c},$$

and therefore $$\frac{a-b}{a}=\frac{b-c}{c}.$$

Now, each of these latter ratios is equal to $\frac{a-c}{a+c}$ and $\frac{a+c-2b}{a-c}$, which are formed by adding and subtracting the antecedents and consequents; therefore

$$\frac{a-c}{a+c}=\frac{a+c-2b}{a-c}.$$

Exercises, LXXVI.

1. Find two Harmonic means between 3 and 4.

2. Insert four Harmonic means between 2 and 12.

3. The Harmonic mean between two quantities is $\frac{4}{5}$ of the Geometric, and the Arithmetic mean is 5; find the quantities.

4. If a, b, c be in G. P., shew that $a+b$, $2b$, $b+c$ will be in H. P.

5. If a, b, c be in H. P., then the Harmonic mean between a and b, b, and the Harmonic mean between b and c, will also be in H. P.

6. If a, b, c be in A. P., and a, b, d in H. P., prove that

$$\frac{c}{d}=1-\frac{2}{ab}(a-b)^2.$$

DETERMINANTS.

311. We have already given in Arts. 251, &c., four methods of solving simultaneous equations of the first degree.

In the following Articles it is proposed to present the method of Cross Multiplication in a new form, in which the eliminating multipliers are denoted by one uniform system of notation, called the notation of *determinants*. This method of determinants is recommended above all others, not only for the uniformity of its processes, but for the facility with which it enables us to express the results of those processes.

312. For the sake of symmetry we shall write one of the unknowns x, y, z......in each term of every proposed equation, so that there will be no constant or absolute term.

Thus we shall use the general equation of two unknowns in the form

$$ax+by=0 \text{.....................} (1);$$

of three unknowns in the form

$$ax+by+cz=0 \text{ ....} (2);$$

of four unknowns in the form

$$ax+by+cz+du=0 \text{ ....} (3);$$

and so on.

Equation (1) is reducible to the ordinary form for one unknown by making $y=1$; equation (2) to the ordinary form for two unknowns by making $z=1$; equation (3) to the ordinary form for three unknowns by making $u=1$; and so on.

When all the equations are written in the forms (1), (2), (3), &c., it will be found that, when the number of given equations is one less than the number of unknowns, we can determine the ratios of the unknowns to one another; when the number of equations is equal to the number of unknowns, we can eliminate all the unknowns, and obtain a relation among the coefficients;

and, when the number of equations is greater than the number of unknowns, we can eliminate all the unknowns in more ways than one, and so obtain several independent relations among the coefficients.

We shall apply the method of determinants to the consideration of

I. One and two equations in two unknowns.

II. Two and three equations in three unknowns.

III. Three and four equations in four unknowns.

313. I. Let there be one equation in x, y,

$$a_1x+b_1y=0;$$

$$\text{then } \frac{x}{b_1}=\frac{y}{-a_1};$$

$$\text{and therefore } \frac{x}{y}=-\frac{b_1}{a_1}\text{............}(1).$$

By putting $y=1$ we deduce from (1) the solution of the simple equation in one unknown $a_1x+b_1=0$,

$$x=-\frac{b_1}{a_1}.$$

314. Again, let there be two equations in two unknowns,

$$a_1x+b_1y=0\text{ ....}(1),$$

$$a_2x+b_2y=0\text{ ....}(2).$$

Multiply (1) by b_2 and (2) by $-b_1$, and add:

$$\therefore (a_1b_2-a_2b_1)x=0,$$

$$\text{or, } (a_1b_2-a_2b_1)=0\text{.........}(3).$$

The relation (3) is called the *eliminant* of (1) and (2), and in the notation of determinants is written

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}=0,$$

$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}$ being called a *determinant*, of which a_1, b_1, a_2, b_2 are *constituents*, and a_1b_2, a_2b_1 *elements*.

315. From the preceding definition of a determinant it appears that

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = (ad - bc),$$

and any sign or symbol written before any determinant will be supposed to operate on every element in the same manner as signs and symbols affect every term within the brackets before which they are written.

Thus

$$\begin{vmatrix} 2, & 3 \\ 4, & 5 \end{vmatrix} = 2 \times 5 - 3 \times 4 = -2;$$

$$\begin{vmatrix} 4, & -1 \\ 2, & 5 \end{vmatrix} = 4 \times 5 - 2(-1) = 22;$$

$$-\begin{vmatrix} 3, & 2 \\ 5, & 6 \end{vmatrix} = -(18 - 10) = -8;$$

$$2\begin{vmatrix} 2, & 0 \\ 5, & 3 \end{vmatrix} = 2(6 - 0) = 12.$$

EXAMPLES.

1. The eliminant of $ax - by = 0$, $a^2x + b^2y = 0$ is

$$\begin{vmatrix} a, & -b \\ a^2, & b^2 \end{vmatrix} = 0,$$

that is, $ab^2 + a^2b = 0$;

and therefore $a + b = 0$.

2. Prove

$$a\begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix} - a_2\begin{vmatrix} a_1, & b_1 \\ a_3, & b_3 \end{vmatrix} + a_3\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = 0.$$

Expanding the determinants we obtain

$$a_1(a_2b_3-a_3b_2)-a_2(a_1b_3-a_3b_1)+a_3(a_1b_2-a_2b_1)$$

$$=a_1a_2b_3-a_1a_3b_2-a_2a_1b_3+a_2a_3b_1+a_3a_1b_2-a_3a_2b_1,$$

which vanishes identically.

EXERCISES, LXXVII.

Find the values of

1. $\begin{vmatrix} 5, & 6 \\ 7, & 9 \end{vmatrix}, \quad \begin{vmatrix} 2, & -1 \\ 3, & -5 \end{vmatrix}, \quad -\begin{vmatrix} -4, & 2 \\ 3, & -2 \end{vmatrix}, \quad -\begin{vmatrix} -5, & 2 \\ -6, & 0 \end{vmatrix}.$

2. $5\begin{vmatrix} 2, & 3 \\ 3, & 2 \end{vmatrix}, \quad -\frac{2}{3}\begin{vmatrix} -4, & 3 \\ 5, & -9 \end{vmatrix}, \quad \frac{1}{ab}\begin{vmatrix} ab, & a \\ b, & ab \end{vmatrix}.$

3. Find the eliminant of $5ax+2y=0$, $3bx-y=0$.

4. Find the eliminant of $ax+by=0$, $bx+ay=0$.

5. Prove $a\begin{vmatrix} c, & d \\ e, & f \end{vmatrix}-c\begin{vmatrix} a, & b \\ e, & f \end{vmatrix}+e\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}=0.$

316. II. Let there be the three unknowns x, y z, and the two equations

$$a_1x+b_1y+c_1z=0\text{............}(1),$$

$$a_2x+b_2y+c_2z=0\text{............}(2).$$

To eliminate z multiply (1) by c_2 and (2) by $-c_1$, and add; therefore

$$(a_1c_2-a_2c_1)x+(b_1c_2-b_2c_1)y=0,$$

$$\text{or }\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}\text{.........}(3).$$

In like manner, by eliminating x or y, we can prove each of the ratios (3) equal to

$$\frac{z}{a_1b_2-a_2b_1}.$$

Hence

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1},$$

or, in the notation of determinants,

$$\frac{x}{\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}}=\frac{y}{-\begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}}=\frac{z}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}} \quad\ldots\ldots(4).$$

317. If $z=1$ we obtain from (4) the solution of the equations

$$a_1x+b_1y+c_1=0,$$
$$a_2x+b_2y+c_2=0.$$

For we have

$$\frac{x}{\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}}=\frac{y}{-\begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}}=\frac{1}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}},$$

and therefore

$$x=\frac{\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}}=\frac{b_1c_2-b_2c_1}{a_1b_1-a_2b_1},$$

$$y=\frac{-\begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}}=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1},$$

318. Let there be three unknowns and three equations.

$$a_1x+b_1y+c_1z=0\ldots\ldots\ldots(1),$$
$$a_2x+b_2y+c_2z=0\ldots\ldots\ldots(2),$$
$$a_3x+b_3y+c_3z=0\ldots\ldots\ldots(3).$$

To eliminate y and z multiply (1), (2), and (3) by

$$\begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix}, \; -\begin{vmatrix} b_1, & c_1 \\ b_3, & c_3 \end{vmatrix}, \text{ and } \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix},$$

respectively, and add; therefore

$$\left\{a_1\begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1, & c_1 \\ b_3, & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}\right\} x = 0.$$

Hence

$$a_1\begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1, & c_1 \\ b_3, & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix} = 0 \ldots (4).$$

This relation (4) is called the *eliminant* of (1), (2), and (3), and is written

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0 \; \ldots\ldots\ldots\ldots\ldots (5),$$

the quantity on the left-hand side of the last equation being called a *determinant*, of which a_1, b_1, c_1, &c. are *constituents*, and $a_1b_2c_3$, $a_1b_3c_2$, &c., *elements*.

Thus, according to the preceding definition,

$$\begin{vmatrix} 6, & 2, & 3 \\ 2, & 4, & 5 \\ 3, & 2, & 4 \end{vmatrix} = 6\begin{vmatrix} 4, & 5 \\ 2, & 4 \end{vmatrix} - 2\begin{vmatrix} 2, & 3 \\ 2, & 4 \end{vmatrix} + 3\begin{vmatrix} 2, & 3 \\ 4, & 5 \end{vmatrix}$$
$$= 36 - 4 - 6 = 26;$$

$$\begin{vmatrix} 2, & 1, & 0 \\ 3, & 0, & 4 \\ 7, & 5, & 1 \end{vmatrix} = 2\begin{vmatrix} 0, & 4 \\ 5, & 1 \end{vmatrix} - 3\begin{vmatrix} 1, & 0 \\ 5, & 1 \end{vmatrix} + 7\begin{vmatrix} 1, & 0 \\ 0, & 4 \end{vmatrix}$$
$$= -40 - 3 + 28 = -15.$$

$$\begin{vmatrix} 3, & -1, & 2 \\ 2, & 1, & -3 \\ -4, & 2, & 5 \end{vmatrix} = 3\begin{vmatrix} 1, & -3 \\ 2, & 5 \end{vmatrix} - 2\begin{vmatrix} -1, & 2 \\ 2, & 5 \end{vmatrix} - 4\begin{vmatrix} -1, & 2 \\ 1, & -3 \end{vmatrix}$$
$$= 33 + 18 - 4 = 47.$$

Examples.

1. Find the ratios $x : y : z$ from

$$2x-3y+4z=0,$$

$$x+2y-5z=0.$$

By (4) Art. 316, we have

$$\frac{x}{\begin{vmatrix} -3, & 4 \\ 2, & -5 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} 2, & 4 \\ 1, & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2, & -3 \\ 1, & 2 \end{vmatrix}},$$

which become, on expanding the determinants,

$$\frac{x}{7} = \frac{y}{14} = \frac{z}{7}.$$

Therefore $\qquad x : y : z = 1 : 2 : 1.$

2. If $(b-c)x+(c-a)y+(a-b)z=0$, find $b-c : c-a$ $a-b$ in terms of x, y, z.

Since $(b-c)+(c-a)+(a-b)=0$, identically, we have

$$x(b-c)+y\ (c-a)+z\ (a-b)=0,$$

$$1\times(b-c)+1\times(c-a)+1\times(a-b)=0,$$

and, therefore,

$$\frac{b-c}{\begin{vmatrix} y, & z \\ 1, & 1 \end{vmatrix}} = \frac{c-a}{-\begin{vmatrix} x, & z \\ 1, & 1 \end{vmatrix}} = \frac{a-b}{\begin{vmatrix} x, & y \\ 1, & 1 \end{vmatrix}},$$

Hence $\qquad \dfrac{b-c}{y-z} = \dfrac{c-a}{z-x} = \dfrac{a-b}{x-y}.$

3. Solve the equations $3x-2y+6=0$, $2x+3y=5$.

Here the equations may be written

$$3x-2y+6\times 1=0,$$

$$2x+3y-5\times 1=0,$$

unity being the value of z in this case.

Therefore

$$\frac{x}{\begin{vmatrix} -2, & 6 \\ 3, & -5 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} 3, & 6 \\ 2, & -5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3, & -2 \\ 2, & 3 \end{vmatrix}},$$

$$\text{or, } \frac{x}{-8} = \frac{y}{27} = \frac{1}{13}.$$

Hence $$x = -\frac{8}{13},\ y = \frac{27}{13}.$$

4. Eliminate x, y, z from $x = a(y+z)$, $y = b(z+x)$, $z = c(x+y)$.

Here the equations may be written

$$\begin{aligned} x - ay - az &= 0, \\ -bx + y - bz &= 0, \\ -cx - cy + z &= 0, \end{aligned}$$

and by (5) Art. 318 the eliminant of these is

$$\begin{vmatrix} 1, & -a, & -a \\ -b, & 1, & -b \\ -c, & -c, & 1 \end{vmatrix} = 0.$$

Expanding this we obtain

$$\begin{vmatrix} 1, & -b \\ -c, & 1 \end{vmatrix} + b\begin{vmatrix} -a, & -a \\ -c, & 1 \end{vmatrix} - c\begin{vmatrix} -a, & -a \\ 1, & -b \end{vmatrix} = 0,$$

that is, $$1 - bc + b(-a - ac) - c(ab + a) = 0,$$

$$\text{or, } bc + ca + ab + 2abc = 1.$$

5. Eliminate x and y from

$$\begin{aligned} 2x + y - a &= 0, \\ x - y + b &= 0, \\ 2x - 3y + c &= 0. \end{aligned}$$

Here $z=1$, and $-a$, b, c may be considered the coefficients of unity; therefore the eliminant required is

$$\begin{vmatrix} 2, & 1, & -a \\ 1, & -1, & b \\ 2, & -3, & c \end{vmatrix} = 0.$$

Expanding this we obtain

$$2 \begin{vmatrix} -1, & b \\ -3, & c \end{vmatrix} - \begin{vmatrix} 1, & -a \\ -3, & c \end{vmatrix} + 2 \begin{vmatrix} 1, & -a \\ -1, & b \end{vmatrix} = 0,$$

that is, $2(-c+3b)-(c-3a)+2(b-a)=0$,

or $$a+8b-3c=0.$$

6. Eliminate a, b, and c from

$$ax+by+c=0,$$

$$ax'+by'+c=0,$$

$$ax''+by''+c=0.$$

Here a, b, c take the place of x, y, z in the usual formulas; and, therefore, the constituents of the determinant will be the coefficients of a, b, c in the proposed equations; hence the eliminant is

$$\begin{vmatrix} x, & y, & 1 \\ x', & y', & 1 \\ x'', & y'', & 1 \end{vmatrix} = 0,$$

which on being expanded becomes

$$x \begin{vmatrix} y', & 1 \\ y'', & 1 \end{vmatrix} - x' \begin{vmatrix} y, & 1 \\ y'', & 1 \end{vmatrix} + x'' \begin{vmatrix} y, & 1 \\ y', & 1 \end{vmatrix} = 0,$$

or $$x(y'-y'')+x'(y''-y)+x''(y-y')=0.$$

Exercises, LXXVIII.

Evaluate

1. $\begin{vmatrix} 1, 2, 1 \\ 2, 1, 2 \\ 3, 1, 3 \end{vmatrix}$, $\begin{vmatrix} 2, 1, 3 \\ 3, 1, 2 \\ 1, 2, 3 \end{vmatrix}$, $\begin{vmatrix} 4, -5, 6 \\ -2, \ \ 4, 0 \\ 1, -2, 3 \end{vmatrix}$.

2. $-2\begin{vmatrix} 3, 0, 1 \\ -5, 6, 2 \\ 4, 2, 0 \end{vmatrix}$, $3\begin{vmatrix} 2, 0, 5 \\ 0, 4, 3 \\ 4, 0, 3 \end{vmatrix}$.

3. Prove

$$b_1\begin{vmatrix} b_2, c_2, d_2 \\ b_3, c_3, d_3 \\ b_4, c_4, d_4 \end{vmatrix} - b_2\begin{vmatrix} b_1, c_1, d_1 \\ b_3, c_3, d_3 \\ b_4, c_4, d_4 \end{vmatrix} + b_3\begin{vmatrix} b_1, c_1, d_1 \\ b_2, c_2, d_2 \\ b_4, c_4, d_4 \end{vmatrix}$$

$$-b_4\begin{vmatrix} b_1, c_1, d_1 \\ b_2, c_2, d_2 \\ b_3, c_3, d_3 \end{vmatrix} = 0.$$

4. Prove that the identity in the preceding question is still true when the multipliers of the determinants are changed from b_1, b_2, b_3, b_4 to c_1, c_2, c_3, c_4, or to d_1, d_2, d_3, d_4, respectively.

5. Find $x:y:z$ from $x+y+z=0$, $ax+by+cz=0$.

6. Find $x:y:z$ from $3x-4y=2z$, $7x-9y-7z=0$.

7. If $(b-c)\,x+(c-a)\,y+(a-b)\,z=0$, prove

$$\frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

8. Solve the equations $3x+2y=32$, $20x-3y=1$.

9. Solve the equations $7x+5y=60$, $13x-11y=10$.

10. Eliminate x, y, z from $ax+by=cz$, $a'x+b'y=c'z$ $''x+b''y=c''z$.

11. Find the condition for the co-existence of the equations

$$ax+by=a'x+b'y=a''x+b''y.$$

12. Eliminate x, y, z from $by+cz=0$, $ax+c'z=0$, $a'x+b'y=0$.

319. III. Let there be four unknowns x, y, z, u, and three equations

$$a_1x+b_1y+c_1z+d_1u=0\ldots(1),$$

$$a_2x+b_2y+c_2z+d_2u=0\ldots(2),$$

$$a_3x+b_3y+c_3z+d_3u=0\ldots(3).$$

In order to eliminate z and u, multiply (1), (2), (3) by

$$\begin{vmatrix} c_2, & d_2 \\ c_3, & d_3 \end{vmatrix}, \quad -\begin{vmatrix} c_1, & d_1 \\ c_3, & d_3 \end{vmatrix}, \quad \begin{vmatrix} c_1, & d_1 \\ c_2, & d_2 \end{vmatrix}, \text{ respectively, and add.}$$

Then, since the coefficients of z and u vanish identically, we obtain

$$\left\{a_1\begin{vmatrix} c_2, & d_2 \\ c_3, & d_3 \end{vmatrix} - a_2\begin{vmatrix} c_1, & d_1 \\ c_3, & d_3 \end{vmatrix} + a_3\begin{vmatrix} c_1, & d_1 \\ c_2, & d_2 \end{vmatrix}\right\}x$$

$$+\left\{ b_1 \begin{vmatrix} c_2, & d_2 \\ c_3, & d_3 \end{vmatrix} - b_2 \begin{vmatrix} c_1, & d_1 \\ c_3, & d_3 \end{vmatrix} + b_3 \begin{vmatrix} c_1, & d_1 \\ c_2, & d_2 \end{vmatrix} \right\} y = 0,$$

which may be written in the equivalent form

$$\frac{x}{\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}}$$

In like manner, by eliminating y and u, and y, z, we prove each of the preceding ratios equal to

$$\frac{z}{\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}} = \frac{u}{-\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}}$$

Hence when we have three equations (1), (2), (3), we can determine the ratios $x:y:z:u$ from the formulas

$$\frac{x}{\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}} = \frac{u}{-\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c \end{vmatrix}} \cdots$$

320. If we put $u=1$, we derive from the last formulas the solution of the equations

$$a_1x+b_1y+c_1z+d_1=0,$$

$$a_2x+b_2y+c_2z+d_2=0,$$

$$a_3x+b_3y+c_3z+d_3=0,$$

in the form

$$x = \frac{\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}}{-\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}}, \text{ \&c.}$$

321. Let there be four unknowns x, y, z, u, and four equations

$$a_1x+b_1y+c_1z+d_1u=0 \text{.........}(1),$$

$$a_2x+b_2y+c_2z+d_2u=0 \text{.........}(2),$$

$$a_3x+b_3y+c_3z+d_3u=0 \text{.........}(3),$$

$$a_4x+b_4y+c_4z+d_4u=0 \text{.........}(4).$$

In order to eliminate y, z, and u, multiply (1), (2, (3), (4) by

$$\begin{vmatrix} b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \\ b_4, & c_4, & d_4 \end{vmatrix}, \; -\begin{vmatrix} b_1, & c_1, & d_1 \\ b_3, & c_3, & d_3 \\ b_4, & c_4, & d_4 \end{vmatrix}, \; \begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_4, & c_4, & d_4 \end{vmatrix}, - \begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix},$$

respectively, and add. The coefficients of y, z, u, vanish identically, and we obtain

$$\left\{ a_1 \begin{vmatrix} b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \\ b_4, & c_4, & d_4 \end{vmatrix} - \text{\&c.} \right\} x = 0.$$

and therefore

$$a_1\begin{vmatrix} b_2, c_2, d_2 \\ b_3, c_3, d_3 \\ b_4, c_4, d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1, c_1, d_1 \\ b_3, c_3, d_3 \\ b_4, c_4, d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1, c_1, d_1 \\ b_2, c_2, d_2 \\ b_4, c_4, d_4 \end{vmatrix}$$

$$-a_4\begin{vmatrix} b_1, c_1, d_1 \\ b_2, c_2, d_2 \\ b_3, c_3, d_3 \end{vmatrix} = 0 \ldots\ldots (5),$$

which is the *eliminant* of (1), (2), (3), (4), and is usually written, in the determinant notation,

$$\begin{vmatrix} a_1, b_1, c_1, d_1 \\ a_2, b_2, c_2, d_2 \\ a_3, b_3, c_3, d_3 \\ a_4, b_4, c_4, d_4 \end{vmatrix} = 0 \ldots\ldots\ldots (6),$$

the left-hand member of the last equation being the determinant of which a_1, b_1, c_1, d_1, &c., are *constituents*, and $a_1\ b_2\ c_3\ d_4$ &c., *elements*.

Thus, according to the preceding definition, the value of

$$\begin{vmatrix} 1, 1, 1, 1 \\ 2, 4, 1, 1 \\ 4, 1, 2, 6 \\ 2, 4, 2, 3 \end{vmatrix}$$

is

$$\begin{vmatrix} 4, 1, 1 \\ 1, 2, 6 \\ 4, 2, 3 \end{vmatrix} - 2\begin{vmatrix} 1, 1, 1 \\ 1, 2, 6 \\ 4, 2, 3 \end{vmatrix} + 4\begin{vmatrix} 1, 1, 1, \\ 4, 1, 1, \\ 4, 2, 3, \end{vmatrix} - 2\begin{vmatrix} 1, 1, 1 \\ 4, 1, 1 \\ 1, 2, 6 \end{vmatrix}$$

the value of which will be found to be -15.

In like manner, we have

$$\begin{vmatrix} 5, & -10, & 11, & 0 \\ -32, & -35, & 34, & 0 \\ 11, & 12, & -11, & 2 \\ 1, & 5, & 3, & 0 \end{vmatrix}$$

$$=5\begin{vmatrix} -35, & 34, & 0 \\ 12, & -11, & 2 \\ 5, & 3, & 0 \end{vmatrix} +32\begin{vmatrix} -10, & 11, & 0 \\ 12, & -11, & 2 \\ 5, & 3, & 0 \end{vmatrix}$$

$$+11\begin{vmatrix} -10, & 11, & 0 \\ -35, & 34, & 0 \\ 5, & 3, & 0 \end{vmatrix} - \begin{vmatrix} -10, & 11, & 0 \\ -35, & 34, & 0 \\ 12, & -11, & 2 \end{vmatrix}$$

the value of which will be found to be 8100.

EXAMPLES.

1. Find the ratios $x:y:z:u$ from

$$2x+3y-4z-10u=0,$$

$$3x-4y+2z-5u=0,$$

$$4x-2y+3z-21u=0.$$

By formula (4) Art. 319 we have

$$\frac{x}{\begin{vmatrix} 3, & -4, & -10 \\ -4, & 2, & -5 \\ -2, & 3, & -21 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} 2, & -4, & -10 \\ 3, & 2, & -5 \\ 4, & 3, & -21 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2, & 3, & -10 \\ 3, & -4, & -5 \\ 4, & -2, & -21 \end{vmatrix}}$$

$$= \frac{u}{-\begin{vmatrix} 2, & 3, & -4 \\ 3, & -4, & 2 \\ 4, & -2, & 3 \end{vmatrix}};$$

Expanding the determinants we get

$$\frac{x}{295}=\frac{y}{236}=\frac{z}{177}=\frac{u}{59}.$$

Hence the required ratios are

$$295:236:177:59,$$

$$\text{or}\quad 5:\ 4:\ 3:\ 1.$$

2. Solve the equations

$$10x-2y+4z-10=0.$$

$$3x+5y+3z-20=0.$$

$$x+3y-2z-21=0.$$

Here $u=1$, and therefore, as in Ex. 1,

$$\frac{x}{\begin{vmatrix} -2, & 4, & -10 \\ 5, & 3, & -20 \\ 3, & -2, & -21 \end{vmatrix}}=\frac{y}{-\begin{vmatrix} 10, & 4, & -10 \\ 3, & 3, & -20 \\ 1, & -2, & -21 \end{vmatrix}}=\frac{z}{\begin{vmatrix} 10, & -2, & -10 \\ 3, & 5, & -20 \\ 1, & 3, & -21 \end{vmatrix}}$$

$$=\frac{1}{-\begin{vmatrix} 10, & -2, & 4 \\ 3, & 5, & 3 \\ 1, & 3, & -2 \end{vmatrix}}$$

Expanding the determinants we get

$$\frac{x}{576}=\frac{y}{768}=\frac{z}{-576}=\frac{1}{192}.$$

Therefore $$x=\frac{576}{192}=3,$$

$$y=\frac{768}{192}=4,$$

$$z=\frac{-576}{192}=-3.$$

3. If $\dfrac{x}{mb+nc-la}=\dfrac{y}{nc+la-mb}=\dfrac{z}{la+mb-nc}$, find $a:b:c$ in terms of x, y, z, l, m, n.

Let each of the given ratios be equal to λ; then the given equalities may be written

$$-la+mb+nc-\frac{x}{\lambda}=0,$$

$$la-mb+nc-\frac{y}{\lambda}=0,$$

$$la+mb-nc-\frac{z}{\lambda}=0.$$

Here we may consider a, b, c, $\dfrac{1}{\lambda}$ to take the place of x, y, z, u in the standard equations; therefore by (4) Art. 319 we have

$$\frac{a}{\begin{vmatrix} m, & n, & -x \\ -m, & n, & -y \\ m, & -n, & -z \end{vmatrix}}=\frac{b}{-\begin{vmatrix} -l, & n, & -x \\ l, & n, & -y \\ l, & -n, & -z \end{vmatrix}}=\frac{c}{\begin{vmatrix} -l, & m, & -x \\ l, & -m, & -y \\ l, & m, & -z \end{vmatrix}}$$

$$=\frac{\dfrac{1}{\lambda}}{-\begin{vmatrix} -l, & m, & n \\ l, & -m, & n \\ l, & m, & -n \end{vmatrix}}.$$

Expanding the first three determinants we get

$$\frac{a}{-2mn\,(y+z)}=\frac{b}{-2nl\,(z+x)}=\frac{c}{-2lm\,(x+y)},$$

and therefore the required ratios are

$$mn\,(y+z):nl\,(z+x):lm\,(x+y).$$

4. Eliminate x, y, z, u from the equations

$$ax+by+z+u=0,$$

$$x+ay+bz+u=0,$$

$$x+y+az+bu=0,$$

$$bx+y+z+au=0.$$

By formula (6) Art. 321 the eliminant of these equations is

$$\begin{vmatrix} a, & b, & 1, & 1 \\ 1, & a, & b, & 1 \\ 1, & 1, & a, & b \\ b, & 1, & 1, & a \end{vmatrix}=0,$$

which on being expanded becomes

$$a^4-2a^2\,(1+2b)\,+4a\,(1+b^2)-4b+2b^2-b^4=0.$$

5. Eliminate a, b, c, d from

$$ax+by+cz-d=0,$$

$$ax'+by'+cz'-d=0,$$

$$ax''+by''+cz''-d=0,$$

$$ax'''+by'''+cz'''-d=0.$$

Here a, b, c, d take the place of x, y, z, u in the standard equations, and therefore the constituents of the determinant will be the constituents of a, b, c, d in the proposed equations.

Hence the eliminant is

$$\begin{vmatrix} x, & y, & z, & -1 \\ x', & y', & z', & -1 \\ x'', & y'', & z'', & -1 \\ x''', & y''', & z''' & -1 \end{vmatrix} = 0,$$

which may be expanded as before.

6. Find the condition for the co-existence of the equations

$$ax+by+c=a'x+b'y+c'=a''x+b''y+c''=a'''x+b'''y+c'''.$$

Let the common value of these quantities be λ; then the equations may be put in the form

$$ax+by+c-\lambda=0,$$

$$a'x+b'y+c'-\lambda=0,$$

$$a''x+b''y+c''-\lambda=0,$$

$$a'''x+b'''y+c'''-\lambda=0.$$

Here 1 and λ take the place of z and u in the standard equations; and therefore the constituents of the determinant will be the coefficients of x, y, 1, λ in the last equations.

Hence the eliminant is

$$\begin{vmatrix} a, & b, & c, & -1 \\ a', & b', & c', & -1 \\ a'', & b'', & c'', & -1 \\ a''', & b''', & c''', & -1 \end{vmatrix} = 0,$$

which on being expanded gives the required condition.

EXERCISES, LXXIX.

1. Evaluate

$$\begin{vmatrix} 1, & 1, & 1, & 4 \\ 2, & 4, & 1, & 8 \\ 4, & 1, & 2, & 13 \\ 2, & 4, & 2, & 11 \end{vmatrix}, \quad \begin{vmatrix} 5, & -10, & 11, & 0 \\ -10, & -11, & 12, & 4 \\ 11, & 12, & -11, & 2 \\ 0, & 4, & 2, & -6 \end{vmatrix}, \quad \begin{vmatrix} 7, & -2, & 0, & 5 \\ -2, & 6, & -2, & 2 \\ 0, & -2, & 5, & 3 \\ 5, & 2, & 3, & 4 \end{vmatrix}.$$

2. Prove

$$\begin{vmatrix} (b+c)^3, & a^3, & a^2 \\ b^2, & (c+a)^3, & b^3 \\ c^2, & c^3, & (a+b)^3 \end{vmatrix} = 2abc(a+b+c)^3.$$

3. Prove

$$\begin{vmatrix} 0, & c, & b, & d \\ c, & 0, & a, & e \\ b, & a, & 0, & f \\ d, & e, & f, & 0 \end{vmatrix} = a^2d^2+b^3e^3+c^2f^2-2abde-2bcef-2adcf.$$

4. Prove

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & z^3, & y^2 \\ 1, & z^3, & 0, & x^2 \\ 1, & y^2, & x^2, & 0 \end{vmatrix} = \begin{vmatrix} 0, & x, & y, & z \\ x, & 0, & z, & y \\ y, & z, & 0, & x \\ z, & y, & x, & 0 \end{vmatrix}.$$

5. Find the ratios $x : y : z : u$ from the equations

$$4x-3y+z+17u=0,\ 9x+7y-5z+34u=0,$$
$$11x-2y+7z-8u=0.$$

6. Solve the equations

$$x-y+z=3,\ 3x+y-z=13,\ x-y-z=-1.$$

7. Solve the equations

$$y+z=5,\ z+x=4,\ x+y=3.$$

8. If $bz+cy=cx+az=ay+bx=\lambda$, prove $x:y:z:\lambda$ equal to

$$a(a-b-c):b(b-c-a):c(c-a-b):-2abc.$$

9. If

$$\begin{aligned} x&=by+cz+du,\\ y&=ax+cz+du,\\ z&=ax+by+du,\\ u&=ax+by+cz, \end{aligned}$$

prove

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=1.$$

ANSWERS.

I.

1. a. 2. $a+b$. 3. $a+b+c$. 4. xy.

II.

1. $a-c$. 2. $-z$. 3. $a-d$. 4. $a+c+e$.

III.

1. $a+b+c=a+(b+c)=a-(-b-c)$.
$a+b-c=a+(b-c)=a-(-b+c)$.
$a-b+c=a+(-b+c)=a-(b-c)$.
$a-b-c=a+(-b-c)=a-(b+c)$.
$a-b+c-d=a+(-b+c)-d=a-(b-c)-d$.

2. $x-y+z-1=-(-x+y)-(-z+1)$.
$-x-y+z-1=-(x+y)-(-z+1)$.
$-x+y+z+1=-(x-y)-(-z-1)$.
$x+y-z-1=-(-x-y)-(z+1)$.

IV.

1. $2a-x(y+z)$. 2. $a^2-b(b-c)$. 3. $a-2c(b-1)-d$.
4. $x-\frac{1}{2}(y+z)$. 5. $2a-\frac{2}{3}p(q-p)$.

V.

1. $18a^3$. 2. $-11a$. 3. $6x$. 4. $x+4y$. 5. 0.
6. x^2-5x. 7. a. 8. $a+\frac{2}{3}b-\frac{1}{4}c$. 9. a^2x+3a^3.

10. $6ab+8ax^2-9a^2x$. 11. $\frac{1}{6}x^2-\frac{1}{12}xy+\frac{1}{20}y^2$.
12. $\frac{35}{12}x-\frac{4}{3}y$. 13. a^2+b^2. 14. $\frac{5}{12}a+\frac{5}{12}b+\frac{5}{12}c$.

VI.

1. $a-7b$. 2. $x-4y+2z$. 3. $2x+2z$. 4. $-4a-4b+4c$.

5. $3x^4-x^3-14x+18$. 6. $2a^2+6b^2-5c^2-16ab+12bc+2ca$. 7. $2b$. 8. $\frac{1}{2}a+\frac{3}{2}b$. 9. x^2-x+1. 10. $\frac{1}{6}yz-\frac{1}{12}zx+\frac{9}{20}xy$.

VII.

1. $a-\{b-(c-d)\}$. 2. $a-\{-b+(c-d)\}$.

3. $a-\{b+(c+d)\}$. 4. $a-\{(b-c)-d\}$.

5. $a-\{-(b-c)+d\}$. 6. $a-\{(b-c)+d\}$.

7. $a-[b+\{-c+(d-e)\}]$.

VIII.

1. y. 2. $a-4b+1$. 3. $2x$. 4. $3y+4z$.

5. $-a-2x$. 6. $2a-b$. 7. c. 8. $-4a$.

9. $9+3x$. 10. $c-e$. 11. $-a-10b-2c$.

IX.

1. $ax+bx+ay-by$. 2. $a-5b$. 3. $3a-6b+6c+6d$.

4. a. 5. $-11a+b$. 6. $2a+4by$.

7. $13a-20b+24c$. 8. $-\frac{1}{6}a+\frac{1}{6}b+c$.

X.

1. $28abc$. 2. $+70x^6$. 3. $-\frac{1}{6}x^2y^2z^2$. 4. $-12y^{-1}$.
5. $10ab$. 6. $12xy$. 7. $-2a^2b$.

XI.

1. $-8x^3+12x^2+4x$. 2. $3x^5+\frac{1}{2}x^3-\frac{3}{2}x^2$.

3. $6x^4y-8x^2y^3+10x^2yz^2$. 4. $6x^{\frac{3}{4}}-3x^{\frac{1}{2}}+6x^{\frac{1}{4}}$.

5. $xy^{-1}-1+x^{-1}y$. 6. $\frac{3}{2}-x^{\frac{1}{4}}+3x^{\frac{1}{2}}$.

XII.

1. $8x^3-2x^2+x-3$. 2. x^4-9x^2+6x-1. 3. x^3+y^3.

4. $a^8+a^4b^4+b^8$. 5. x^5+y^5. 6. $1-2x-31x^2+72x^3-30x^4$.

7. $2x^5-18x^4+39x^3-25x^2+x+1$.

8. $4x^6-5x^5+8x^4-10x^3-8x^2-5x-4$.

9. $x^3+y^3+3xy-2x-2y+1$.

10. $x^2-4y^2+12yz-9z^2$.

11. $a^3+b^3+c^3-3abc$. 12. $a^4-2a^2b^2+b^4+4abc^2-c^4$.

13. $x-y$. 14. $x^{\frac{3}{2}}-y^{\frac{3}{2}}$. 15. $a-b$. 16. a^2+b^2.

17. $x^2+2x^{\frac{3}{2}}+x-4$. 18. x^4-2+x^{-4}. 19. x^4+1+x^{-4}.

20. $a^{-1}-1$. 21. $a^2-3a^{\frac{2}{3}}+3a^{-\frac{2}{3}}-a^{-2}$.

22. $a^2+2a^{\frac{3}{2}}b^{\frac{1}{2}}+ab-x^{\frac{2}{3}}y^{\frac{4}{3}}$.

23. $x^{\frac{5}{2}}+x^{\frac{3}{2}}y-xy^{\frac{3}{2}}-y^{\frac{5}{2}}$.

24. $\frac{4}{3}x^2-\frac{1}{3}xy-\frac{7}{6}x-\frac{1}{2}y^2+\frac{11}{4}y-3$.

25. $\frac{1}{6}x^2+\frac{5}{36}x-\frac{1}{3}+\frac{5}{6}x^{-1}-x^{-2}$.

26. $x^{3p}-y^{8p}$.

27. $a^{m+n}-4a^{m+n-2}x^2+6a^{m+n-3}x^3-9a^{m+n-4}x^4$.

28. $x^{\frac{r}{2}}+2x^{\frac{r}{2}}y^{2p}+9y^{4p}$.

XIII.

1. x^3+x^2-x-1. 2. $2x^5-x^4-6x+5$.

3. $36x^6-x^2+2x-1$. 4. $24x^9+4x^4-3x^3+2x^2+1$.

5. $21x^8+14x^7-49x^6-8x^5-10x^4+41x^3-x^2-14x+2$.

XIV.

1. $1+2x+3x^2+2x^3+x^4$. 2. $1-2x+3x^2-2x^3+x^4$.

3. $1-2x-x^2+2x^3+x^4$. 4. $1-6x+13x^2-12x^3+4x^4$.

5. $x^4-x^3+\frac{17}{4}x^2-2x+4$.

6. $x^4-4x^2+6-4x^{-2}+x^{-4}$. 7. $x^3-2x^2+2x^{\frac{3}{2}}+x$
$-2x^{\frac{1}{2}}+1$.

8. $\frac{1}{4}x-\frac{1}{3}x^{\frac{5}{6}}+\frac{1}{9}x^{\frac{2}{3}}-x^{\frac{1}{2}}+\frac{2}{3}x^{\frac{1}{3}}+1$.

9. $a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$.

10. $24x+48x^3$.

11. $4ab+4ad+4bc+4cd$.

XV.

1. a^2-9b^2. 2. a^4-b^4. 3. $a^{\frac{4}{3}}-b^{\frac{4}{3}}$.

4. $a-b$. 5. $16a^{\frac{1}{2}}-25b^{\frac{2}{5}}$. 6. $\frac{1}{49}a^4-\frac{1}{81}b^2$.

7. $\frac{a^4}{b^2}-\frac{c^2}{d^4}$. 8. $4x^2+y^2+4xy-z^2$.

9. $25a^2-9b^2+6bc-c^2$. 10. $4a^4-9b^4-24b^2c^2-16c^4$.

11. $x^2-2xy+y^2-z^2+2z-1$.

12. $4x^2+9y^2+z^2-12xy+4xz-6yz-1$.

13. $4a^4-9b^2-c^2-16d^2+6bc-24bd+8cd$.

XVI.

1. $8x^3+27y^3$. 2. x^3-8y^3. 3. a^6-b^3.

4. $a^{\frac{3}{2}}+b^{\frac{3}{2}}$. 5. $8x^{\frac{3}{2}}-27y^{\frac{3}{2}}$. 6. $x^{3p}+y^{3q}$.

7. $a^3x^{-3}+a^{-3}x^3$.

XVII.

1. $-2x$. 2. $-3xy^2$. 3. $-x$. 4. $-x^2yz$.

5. $-2a$. 6. $-5xy$. 7. $a^{\frac{1}{6}}b^{\frac{1}{12}}c^{\frac{1}{20}}$.

XVIII.

1. $6x^2-2x+3$. 2. $-3ax^2+5bxy-7cy^2$.

3. mx^3y-nxy^3. 4. ax^m-bx^n.

5. $-5x^3+6y^3$. 6. mx^m-ny^n-1.

XIX.

1. $1-2x+x^2$. 2. $5x^2-4x+2$. 3. x^2-x+1.

4. x^2+x+3. 5. $3x^2-2x+4$. 6. x^2-3x+7.

7. $x^4+x^3+x^2+x+1$. 8. x^2-2x+2. 9. x^2-5x+6.

10. $1-3x+2x^2-x^3$. 11. x^2-1. 12. $x^4+2x^3+3x^2$
$+2x+1$.

13. $\frac{3}{2}x^3-5x^2+\frac{1}{4}x+9$. 14. x^2-1+x^{-2}. 15. $x-2$
$+x^{-1}$.

16. $y^4+y^2+1+y^{-2}+y^{-4}$. 17. $x+2+x^{-1}$.

18. $x-4x^{\frac{1}{2}}+8$. 19. $x-3x^{\frac{1}{2}}-1$. 20. $x^2+2xy+3y^2$.

21. $a^3+2a^2b+2ab^2+b^3$. 22. a^2-3b^2+1. 23. x^2-mx
$+n$.

24. $a+b+c$. 25. x^2-mx+m^2-n. 26. $a^2-2ab+3b^2$.

27. $x^2+x\,(y+1)+y^2-y+1$.

28. $x-c$. 29. $x^2+y^2+z^2-xy-xz-yz$.

30. $x^{\frac{3}{2}}+xy^{\frac{1}{4}}+x^{\frac{1}{2}}y^{\frac{1}{2}}+y^{\frac{3}{4}}$.

31. $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$.

32. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$. 33. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$.

34. $a+2a^{\frac{1}{2}}b^{-\frac{1}{2}}+3b^{-1}$. 35. $x^{\frac{3}{2}}-xy^{-\frac{1}{2}}+x^{\frac{1}{2}}y^{-1}$.

36. $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{-\frac{1}{6}}+b^{-\frac{1}{3}}$. 37. $x+2a^{\frac{1}{2}}x^{\frac{3}{4}}+3ax^{\frac{1}{2}}+2a^{\frac{3}{2}}x^{\frac{1}{4}}$
$+a^2$.

38. $x^{\frac{1}{2}}-2x^{\frac{1}{4}}y^{-\frac{1}{8}}+y^{-\frac{1}{4}}$.

XX.

1. $2x^2-3x+\frac{11}{2}$, $-\frac{7}{2}$. 2. $x-a$, $2a^2$.

3. x^2-ax+a^2, $-2a^3$. 4. x^2-x+4, $-3x-4$.

5. $2x^2+3$, $-5x^2-3x-3$.

6. $x^3+x^2y+xy^2+y^3$, $2y^5$.

7. $x-2a$, $9a^2x+2a^3$.

XXI.

1. $5x^2+11x+30$. 2. x^2+2x+1. 3. x^3-3x^2+3x-1

4. $20x^4-3x^2+2$. 5. $17x^4+19x^3+38x^2+76x+152$.

6. x^5+x^3-x-1. 7. $4x^{\text{[illegible]}}-3x^2+2x+2$.

8. $11x^2-2x+77$. 9. $5x^2-12x+12$, $12x-72$.

10. $4x^3+2x^2-2x+3$, 15.

11. $7x^4-2x^3+x^2-3x+9$, -27.

12. $7x^2+14x+4$, $-34x-11$.

13. $10x^3+5x^2+1$, $10x+10$.

14. $5x^2+10x+5$, $-5x^2-10x+27$.

15. $8x-56$, $16x^2+56$. 16. $10x^3$, $10x^4-100$.

17. $3x^3+11x^2+19x+45$, $90x-8$.

18. $9x^5-18x$, $4x^6+18x+1$.

XXII.

1. 2. 2. 7. 3. $\frac{1}{3}$. 4. 1. 5. b^2-a^2. 6. $-2z$.

XXIII.

1. 1. 2. 47. 3. 36. 4. -205. 5. 0. 6. 0.

7. $2a^9$. 8. 1.

XXIV.

1. 20. 2. 300. 3. -535. 4. 1. 5. 10. 6. -1.

7. -800. 8. $7\frac{2}{3}$. 9. -8.

XXV.

1. $x^2-2xy+y^2$. 2. $4x^2-12xy+9y^2$.

3. $1-2x+2y-2xy+x^2+y^2$.

4. $4a^2-12ab+9b^2-4ac+6bc+c^2$.

5. $y^2z^2+2xyz^2+z^2x^2+2xy^2z+2x^2yz+x^2y^2$.

6. $4x^2-4x+5-2x^{-1}+x^{-2}$.

7. $x^4+2ax^2+a^2-2ax-2a^2x^{-1}+a^2x^{-2}$.

8. $x^2-2xy+y^2+2xz-2yz+z^2-2x+2y-2z+1$.

9. $a^4-2a^2b+b^2+2a^2c^2-2bc^2+c^4-2a^2d+2bd-2c^2d+d^2$.

10. $4x^4-12x^3+13x^2-10x+7-2x^{-1}+x^{-2}$.

XXVI.

1. $y^3+3y^2z+3yz^2+z^3$. 2. $8a^3-36a^2b+54ab^2-27b^3$.

3. $8a^6+36a^4+54a^2+27$. 4. $x^6-6x^4+12x^2-8$.

5. $8x^3+36x^2y^2+54xy^4+27y^6$.

6. $1-3a+6a^2-7a^3+6a^4-3a^5+a^6$.

7. $x^3-3x^2+6x-7+6x^{-1}-3x^{-2}+x^{-3}$.

8. $x^6+6x^4+9x^2-4-9x^{-2}+6x^{-4}-x^{-6}$.

9. $1-x^3+y^3-z^3-3x+3y-3z+3x^2+3x^2y-3x^2z+3y^2-3xy^2-3y^2z+3z^2-3xz^2+3yz^2-6xy+6xz-6yz+6xyz$.

10. $b^3c^3+c^3a^3+a^3b^3+3ab^2c^3+3ab^3c^2+3a^3bc^2+3a^2bc^3+3a^2b^3c+3a^3b^2c+6a^2b^2c^2$.

XXVII.

1. $x^3+3x^2y+3xy^2+y^3$. 2. $x^3-3x^2y+3xy^2-y^3$.

3. $a^3-3a^2b^2+3ab^4-b^6$.

4. $x^3-3x^2y+3xy^2-y^3-3x^2z+6xyz-3y^2z+3xz^2-3yz^2-z^3$.

5. $x^6+3x^4y+3x^2y^2+y^3-3x^4z^2-6x^2yz^2-3y^2z^2+3x^2z^4+3yz^4-z^6$.

6. $1+3x-5x^3+3x^5-x^6$.

XXVIII.

1. $1-4a+6a^2-4a^3+a^4$.

2. $x^4-4x^3y+6x^2y^2-4xy^3+y^4$.

3. $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$.

4. $a^8-4a^6b+4a^6c+6a^4b^2+6a^4c^2-12a^4bc-4a^2b^3+4$
$+12a^2b^2c-12a^2bc^2-4bc^3-4b^3c+6b^2c^2+b^4+c^4$.

XXIX.

1. $1+x+x^2+x^3+$ &c.

2. $1-x^2+x^4-x^6+$ &c.

3. $1-x+x^3-x^4+$ &c.

4. $1+x+x^2+2x^3+$ &c.

5. $1+x+2x^2-2x^3+$ &c.

6. $1-x+2x^2-3x^3+$ &c.

XXX.

1. $2x^2+a$. 2. $2a-3b$. 3. $x-\frac{p}{2}$.

4. a^2+a+1. 5. $\frac{3}{2}a^2+\frac{2}{3}b^2$. 6. $2x^{-2}+3x^{-1}$.

7. $2x^{\frac{1}{2}}-1$. 8. $x+2-x^{-1}$. 9. $3a-2b+4c$.

10. $6x^2-3x+\frac{2}{3}$. 11. x^2+4+4x^{-2}. 12. x^n-p.

13. $x^{\frac{1}{2}}-x^{-\frac{1}{2}}+x^{-1}$. 14. $x^n+px^{\frac{n}{2}}+q$. 15. $5a^4+2ab^3$
$-3b^4$.

XXXI.

1. $1+2x$. 2. $3x-2$. 3. $2a-7x$. 4. $3x+2x^{-1}$.
5. $2x^2-x+1$. 6. $3x^2-2ax+a^2$.

XXXII.

1. $4ax^2$. 2. $3ax$. 3. $5x^2y^3z^2$. 4. $abuv$.
5. $4ab$. 6. $xyuv$.

XXXIII.

1. $x+1$. 2. $x+3$. 3. $x-1$. 4. $x-1$.
5. $x-1$. 6. x^2-1. 7. $x-2$. 8. $x+9$.
9. x^2-4x+3. 10. $x-7$. 11. $x+6$. 12. $x-1$.
13. x^2-2x+1. 14. $5x^2-1$. 15. $2x+5$. 16. $x+1$.
17. $x-1$. 18. x^2-a^2. 19. x^2+y. 20. x^2+y^2.
21. $x+3y$. 22. x^2-y^2.

XXXIV.

1. $2(x-1)$. 2. $3x(x^2+xy+y^2)$. 3. $ax(x-y)$.
4. $ab(2a^3+3a^2b-ab^2+b^3)$.

XXXV.

1. $x+6$. 2. $3x-2$. 3. $3a-2b$. 4. $x-2$.

5. $x+y$. 6. $x-3$.

XXXVI.

1. $36a^2b^2c^3$. 2. $105ax^6y^4z$. 3. $48xy(x^3-y^3)$.

4. $2(4x+1)(2x^3-5x^2+5x-3)$.

5. $(x-1)(x+1)(x^2+1)$. 6. $(2x-1)(3x-1)(4x+1)$.

7. $(x-1)(2x-1)(2x+1)(3x-2)$.

XXXVII.

1. $24x^2y^2z^3$. 2. $(x+6)(x+5)(x-5)(x-7)$.

3. $(x+1)(x-1)(7x+2)(7x-2)$.

4. $5(3a-2b)(4a+5b)(5a+4b)(a+b)$.

5. $(x^2-4)(4x^2-1)(x^2-2x+4)$.

6. $(7x-2y)^2(7x+2y)(3x-2y)(3x+2y)$.

XXXVIII.

1. $\dfrac{x^3y^2}{ay^2-bx}$. 2. $\dfrac{bx-ay}{a^2bx+ab^2y}$. 3. $\dfrac{3x^4-12x^2+4x}{6x^5-12x^3-18}$.

4. $\dfrac{x^2-1}{x^{\frac{5}{3}}+x^{\frac{1}{3}}+x}$ 5. $\dfrac{a^2x^2-abx^{\frac{1}{2}}+ax}{x^3-ab-ax}$.

XXXIX.

1. $\dfrac{2a}{5b}$. 2. $\dfrac{3y}{4xz}$. 3. $\dfrac{3cz}{4axy}$.

4. $\dfrac{ay-bz}{yz}$. 5. $\dfrac{c}{a-b}$. 6. $\dfrac{x+1}{x-1}$.

7. $\dfrac{3x-5y}{2x+y}$. 8. $\dfrac{2x+1}{x-2}$. 9. $\dfrac{x-y}{x+y}$.

10. $\dfrac{p+mq}{p^2+q^2}$. 11. $\dfrac{a-2b}{a+b-c}$. 12. $\dfrac{x^2-y^2}{x^6+y}$.

13. $\dfrac{4x+3y}{3x^2+y^2}$.

XL.

1. $\dfrac{a}{abc}$, $\dfrac{c}{abc}$. 2. $\dfrac{x}{xyz}$, $\dfrac{y}{xyz}$, $\dfrac{z}{xyz}$. 3. $\dfrac{a^2}{abc}$, $\dfrac{b^2}{abc}$, $\dfrac{c^2}{abc}$.

4. $\dfrac{x+1}{x^2-1}$, $\dfrac{2(x-1)}{x^2-1}$. 5. $\dfrac{x-1}{x^3-x^2-x+1}$, $\dfrac{x+1}{x^3-x^2-x+1}$.

6. $\frac{3x(x^2-1)}{x^4-1}$, $\frac{5x^2}{x^4-1}$. 7. $\frac{2(x^2-1)}{x^3-x}$, $\frac{3x(x+1)}{x^3-x}$, $\frac{5x}{x^3-x}$.

8. $\frac{c-b}{(b-c)(c-a)(a-b)}$, $\frac{a-c}{(b-c)(c-a)(a-b)}$,

$\frac{b-a}{(b-c)(c-a)(a-b)}$.

XLI.

1. $\frac{2x^3-3y^3}{6xy}$. 2. $\frac{3ad-2b^2}{12bcd}$. 3. $-\frac{1}{x^2-5x+6}$.

4. $\frac{-2x^2-10x-6}{15x^2+22x+8}$. 5. $\frac{16+6y}{4-9y^2}$. 6. $\frac{2y^2}{x^4-y^4}$.

7. $\frac{2+2a^2}{1+a^2+a^4}$. 8. $\frac{2x^3}{1+x^2+x^4}$. 9. $\frac{2xy}{x^4+x^2y^2+y^4}$.

10. $\frac{x^3+10x^2+23x+12}{(x+1)(x+2)(x+3)}$. 11. $\frac{8x+5}{(2x+1)(3x+2)(4x+3)}$.

12. $\frac{2}{a+b}$. 13. $\frac{x-3x^2+x^3}{(1-x)^3}$. 14. $\frac{85a-20b}{84}$.

15. $\frac{2a}{a+b}$. 16. $\frac{2(1+x^2)}{x(1-x^2)}$.

17. $\frac{a^4+2a^3b-2a^2b^2+2ab^3+b^4}{a^4-b^4}$. 18. 0.

19. $\frac{x^2}{(x+1)(x+2)(x+3)}$. 20. $\frac{1+x+x^2}{(1+x)(1+x^2)(1-x)^2}$.

21. 2. 22. $\frac{x}{(x-a)(x-b)(x-c)}$.

XLII.

1. $\frac{x^2+y^2-2xy}{xy}$. 2. $\frac{abcx^3+cx+2b}{bcx^2}$. 3. $\frac{1-xy}{x-y}$.

4. $\frac{2x^2+x-4}{x+1}$. 5. $\frac{6x^2+6x}{2x+3}$. 6. $\frac{x^3-xy}{x-y}$.

7. $1-\frac{3}{2xy}$. 8. $\frac{x^2-1}{2y^2}-1$. 9. $x-3+\frac{3x-7}{x^2-3}$.

10. $3-\frac{a-2}{2a^2-a+1}$. 11. $a^2-\frac{a^2+a-5}{a^3+1}$.

12. $x^2-\frac{3x^n-1}{x^{2n}+1}$.

XLIII.

1. $\frac{xy}{1-x^2}$. 2. $\frac{a+x}{c(a-x)}$. 3. $\frac{a^2+b^2}{a}$.

4. $\frac{m^2}{3(m-n)n}$. 5. $\frac{c}{2b(a+x)}$. 6. $\frac{1}{(m-n)^2}$.

7. 1. 8. 1. 9. $2(a+b+c)$. 10. 1. 11. 2.

12. $\frac{x+y}{x-y}$. 13. $\frac{1+x}{x}$. 14. $\frac{(a-b)^2}{a^2+b^2}$.

XLIV.

1. $\dfrac{x-y}{x+y}$. 2. $\dfrac{a^2}{b(a-b)}$. 3. $\dfrac{x^2+xy+y^2}{x^2+y^2}$.

4. $\dfrac{(x^2-xy+y^2)\,(a-b)^2}{(a^2+ab+b^2)\,(x^3-y^3)}$. 5. $\dfrac{(b-c)^2}{(a-b)^2}$.

6. $\dfrac{(x-1)\,(x-2)}{x^2}$. 7. $\dfrac{x^2+y^2}{x}$. 8. $\dfrac{xy}{x^2+y^2}$.

9. $\dfrac{a^4+a^2b^2-ab^3}{b^4+b^2a^2-ba^3}$.

XLV.

1. $\sqrt{18}$, $\sqrt[3]{56}$, $\sqrt[4]{405}$. 2. $2\sqrt[3]{6}$, $3\sqrt[3]{4}$.

3. $\sqrt[6]{125}$, $\sqrt[6]{27}$. 4. $\sqrt[9]{8}$, $\sqrt[9]{27}$, $\sqrt[9]{125}$.

5. $\sqrt[6]{3}$, $\sqrt[4]{5}$.

XLVI.

1. $7\sqrt{2}$. 2. $3\sqrt{3}$. 3. $9\sqrt[3]{4}$. 4. $36\sqrt{3}$.

5. $5\sqrt[3]{2}$. 6. $\sqrt{3}$.

XLVII.

1. $6\sqrt{15}$. 2. $6\sqrt[3]{28}$. 3. $\sqrt[6]{4500}$. 4. $10\sqrt[12]{432}$.

5. $\sqrt[12]{72}$. 6. $\sqrt{15}-3\sqrt{2}$. 7. $\frac{1}{6}$. 8. $2\sqrt{2}-2\sqrt{3}$ $+2$.

9. $2+\sqrt[6]{72}-\sqrt[4]{12}-\sqrt[12]{2187}$.

10. $\sqrt[16]{2^5 . 3^{10}}$, $\sqrt[16]{2^{10} . 3^5}$, $\sqrt[4]{8}$, $\sqrt[8]{32}$.

XLVIII.

1. $\sqrt[15]{9}$. 2. $\frac{1}{2}\sqrt[6]{5}$. 3. $\sqrt{6}$. 4. $\sqrt[3]{4}$.

5. $\sqrt[20]{3}$ 6. $\sqrt[15]{2^7}$.

XLIX.

1. $\frac{5\sqrt{7}}{7}$, $\frac{2\sqrt[3]{25}}{5}$, $\frac{\sqrt[5]{6^4}}{6}$. 2. $\frac{\sqrt{7}+\sqrt{3}}{2}$.

3. $\frac{1}{7}(1+2\sqrt{3})(3+\sqrt{2})$. 4. $\frac{3-\sqrt{5}}{2}$.

5. $\frac{(3-\sqrt{7})^2}{2}$. 6. $\frac{(4+\sqrt{2})(4-\sqrt{3})}{13}$.

7. $(\sqrt{8}+\sqrt{12})(\sqrt{3}+\sqrt{2})$.

8. $\frac{(2+\sqrt{3}-\sqrt{5})(2\sqrt{3}-1)}{22}$.

9. $-\frac{(3-\sqrt{2}-\sqrt{3})\,(4+3\sqrt{2})}{4}$.

10. $-\frac{2\,(1-\sqrt{5}+\sqrt{3})\,(3+2\sqrt{5})}{11}$.

11. $\frac{(\sqrt{2}+\sqrt{5}-\sqrt{7})\,\sqrt{10}}{4}$.

12. $-\frac{(\sqrt{2}-\sqrt{5}+\sqrt{7})\,\sqrt{10}}{10}$.

L.

1. 5. 2. 3. 3. 3. 4. $\frac{1}{10}$. 5. 3. 6. $2\frac{1}{3}$.
7. 2. 8. 1. 9. $2\frac{1}{3}$. 10. 3. 11. -2
12. 2. 13. 7. 14. $\frac{c}{a-b}$. 15. $\frac{n-b}{a-m}$. 16. $\frac{2ab}{a+b}$.
17. $\frac{a^2+ab+b^2}{a+b}$. 18. $\frac{ab-pq}{a+b+p+q}$.
19. $\frac{2\,(a^2+ab+b^2)}{3\,(a+b)}$. 20. $\frac{1}{2}\,(a+b)$.

LI.

1. 105. 2. $3\frac{1}{3}$. 3. 72. 4. 6. 5. 11.
6. 4. 7. 17. 8. 7. 9. 10. 10. 5. 11. 5.

12. $\frac{ad}{bc}$. 13. $\frac{a-b}{a+b}$. 14. 1. 15. $\frac{3}{2}$.

16. $\frac{2a^2+2b^2}{a+b}$. 17. 2. 18. $\frac{50}{29}$. 19. $\frac{3}{2}$.

20. $b-a$. 21. $\frac{2ab}{a+b}$. 22. $\frac{a^2}{b-a}$. 23. $\frac{2a-3}{4-3a}$.

24. $\frac{2ab}{a+b}$.

LII.

1. -4. 2. $\frac{11}{6}$. 3. $4\frac{4}{23}$. 4. $4\frac{7}{9}$. 5. 21.

6. 5. 7. $5\frac{2}{3}$. 8. $\frac{25}{88}$. 9. 4. 10. $\frac{c^2-b^2}{2c+a}$.

11. $(\sqrt{a}+\sqrt{b})^2$. 12. $\frac{1}{3}$. 13. $\frac{4}{7}$. 14. $\frac{9a}{16}$.

15. $\frac{289}{864}$. 16. $25x^2-50x+9=0$. 17. $x^2=162$. 18. $x^2-2ax+a^2=0$.

LIII.

1. 7. 2. 12, 18. 3. 4, 7. 4. 6. 5. $\frac{3}{2}$.

6. £28 10s., £5 14s. 7. 23s., 16s. 8. £12,000.

9. £112 10s., £75. 10. $13\frac{1}{2}$ miles. 11. 30 gallons.

12. $\dfrac{ap+bq+cr}{a+b+c}$. 13. $28\frac{1}{8}$ min. 14. 12 miles.

15. $4\frac{2}{7}$ miles. 16. $3\frac{1}{2}$ min. 17. 20, 4. 18. 18, 12.

19. 35. 20. 22 crowns, 180 half-guineas.

21. $5\frac{5}{11}$ min. past 7. 22. 4 seconds. 23. At $\frac{12m}{11}$ hours, where m lies between 1 and 11, inclusive.

24. 72 leaps, while the hare takes 108.

25. $a\cdot\dfrac{t'-t}{t'+t}$. 26. 3 miles, 12 miles.

27. 55 sixpences, 59 shillings, 4 sovereigns.

28. 21 miles per hour, 42 miles per hour.

29. 500. 30. 84. 31. 84. 32. 12 miles per hour.

33. $26\frac{8}{11}$ min. past 2.

34. Son's share £900, daughter's £600, widow's £750.

35. 4000, 1500, 400.

36. $\dfrac{nw+n'w'}{w+w'}$, $\dfrac{sw+s'w'}{w+w'}$, $\dfrac{cw+c'w'}{w+w'}$.

37. 15 cwt. 38. 90 men. 39. One million.

40. £1750 and £3472.

LIV.

1. $1, 2$. 2. $0, 3\frac{1}{2}$. 3. $-3, 5$. 4. $2\frac{1}{2}, -3$.

5. $\frac{1}{6}, -\frac{3}{2}$. 6. $0, \frac{3}{7}$. 7. $2, -2$. 8. $2\sqrt{-1}$,
$-2\sqrt{-1}$.

9. $3, 5$. 10. $-2, -7$. 11. $4, -3$. 12. $4, -5$.

13. $\frac{1}{2}, \frac{1}{3}$. 14. $\frac{1}{5}, -\frac{1}{6}$. 15. $\frac{1}{3}, -\frac{4}{5}$. 16. $-\frac{2}{7}, -\frac{3}{5}$.

17. $2, \frac{1}{2}$. 18. $6, -18$. 19. $3, -\frac{5}{7}$. 20. $6, 0$.

21. $-a, -b$. 22. $\frac{8}{a}(1 \pm \sqrt{2})$. 23. $\frac{\sqrt{n^2-1} \pm n}{a}$

24. $\frac{1 \pm \sqrt{1+a^2}}{a}$ 25. $8, -\frac{8}{9}$. 26. $0, -10$.

27. $\frac{b}{a}, -\frac{d}{c}$. 28. $\frac{4 \pm \sqrt{106}}{10}$. 29. $\frac{-3 \pm \sqrt{93}}{14}$.

30. $14, 2{\cdot}48$. 31. $\frac{2}{5}(7 \pm \sqrt{55})$.

LV.

1. $\pm 2, \pm\sqrt{-6}$. 2. $4, 3\sqrt[3]{3}$. 3. $-1, \dfrac{1\pm\sqrt{-3}}{2}$.

4. 4, 5. 5. $-2, \frac{1}{2}, \dfrac{1\pm\sqrt{10}}{3}$. 6. $2, -3, \dfrac{-1\pm 3\sqrt{5}}{2}$. 7. $1, 1, 1\pm 2\sqrt{15}$. 8. $18\frac{1}{2}, 5$.

9. $3, 4, \frac{1}{3}, \frac{1}{4}$. 10. $6, 3, \frac{1}{6}, \frac{1}{3}$. 11. 7, 9. 12. 3, 5.

13. $10, -\frac{32}{3}, -\frac{1}{3}\pm\frac{1}{6}\sqrt{3631}$

14. $1, -\frac{11}{5}, -\frac{3}{5}\pm\frac{1}{15}\sqrt{491}$. 15. $5, -30$.

16. $\pm\frac{7}{5}$. 17. $5, -11\frac{1}{4}$. 18. ± 1.

19. $\pm 1, \pm\sqrt{-\frac{24}{35}}$.

LVI.

1. 8, 6. 2. 6, 4. 3. 24. 4. $8, 3\frac{1}{2}$. 5. 4, 9.

6. £70. 7. 20 days. 8. 15 and 30 days.

9. 64, 36. 10. 24 in. 11. 44 yards, 110 yards.

12. 4 miles an hour. 13. 12, or 8. 14. 87.

15. 53. 16. 25, 30. 17. 9. 18. 64.

LVII.

1. $(x-2)(x-9)$. 2. $(x+4)(x-13)$.

3. $(2x-1)(3x+4)$. 4. $(7+5x)(1-2x)$.

5. $(11+10x)(3-4x)$. 6. $(5-19x)(13-11x)$.

LVIII.

1. $\frac{11}{4}$. 2. 4. 3. $\frac{1}{13}$. 4. 1 5. $50\frac{7}{32}$.

LIX.

1. 10, 7. 2. 5, 2. 3. 2, -3. 4. 4, 12.

5. 3, 2. 6. $3\frac{1}{2}$, 4. 7. $\frac{1}{4}$, $\frac{2}{5}$. 8. 19, 2.

9. 6, 12. 10. $\frac{1}{4}$, $\frac{1}{5}$. 11. a, b. 12. $\frac{ab}{a+b}, \frac{ab}{a+b}$.

LX.

1. 5, 11. 2. 11, 7. 3. 4, 6. 4. 5, 7.

5. $\frac{2abc}{a^2+b^2}, \frac{a^2-b^2}{a^2+b^2}\cdot c$. 6. 7, 9. 7. $\frac{qc-br}{aq-bp}, \frac{ar-pc}{aq-bp}$.

8. $\frac{a+b}{a-b-2}$, $\frac{a+b}{a-b-2}$.   9. $\frac{c\,(c-b)}{a\,(a-b)}$, $\frac{c\,(c-a)}{b\,(b-a)}$.

LXI.

1. 2, 1.   2. 4, 3.   3. $8\frac{1}{55}$, $9\frac{17}{55}$.   4. 8, 2.

5. $\frac{1}{2}$, $\frac{1}{3}$.   6. $\frac{3}{4}$, $\frac{2}{5}$.   7. a, b.   8. $\frac{ac}{a+b}$, $\frac{bc}{a+b}$.

9. $\frac{1}{a+b}$, 0.   10. a, b.

LXII.

1. 2, 1.   2. 4, -5.   3. $\frac{1}{2}$, $\frac{1}{3}$.   4. $1\frac{1}{2}$, $-2\frac{1}{2}$.

5. 11, -9.   6. $\frac{bp-cn}{bm-an}$, $\frac{cm-ap}{bm-an}$.

7. $\frac{m^2-n^2}{m\,(t-r)-n\,(t+r)}$, $\frac{(m-n)^2}{m\,(t-r)-n\,(t+r)}$.

8. $\frac{(q^2-p^2)\,n-(m^2-n^2)\,p}{nq-mp}$, $\frac{(m^2-n^2)\,q-(q^2-p^2)m}{nq-mp}$.

LXIII.

1. 2, 1, 3.   2. 3, 4, 6.   3. 2, 1, 3.   4. 9, 11, 13.

5. 4, 0, 5. 6. 5, -5, 5. 7. 2, 4, 6. 8. 5, 7, 9.

9. $\frac{1}{2}(b+c-a)$, &c. 10. $\frac{2}{3}(a+b+c)-a$, &c.

11. $\frac{1}{2}(b+c)$, &c. 12. $x=y=z=\frac{abc}{bc+ca+ab}$.

LXIV.

1. 12, 14. 2. 35. 3. $\frac{6}{16}$. 4. 9, 7, 6.

5. 3, 4. 6. $112\frac{1}{2}$ min., 90 min. 7. £141 1s. 6d.

8. 72, 64, 56, 48. 9. A in 3 days, B in 6 days, C in 9 days.

10. 24. 11. $(p+1)n$, $(pq-1)n$, $(q+1)n$.

12. Passed 8, rejected 6, sent back 3.

13. 15, 25. 14. 24, 60, 120. 15. 110, 50. 16. 84.

17. A £75, B £35 2s., C £51 6s. At end each has £53 16s.

18. 60 min., 75 min., 90 min. All together in $24\frac{12}{37}$ min.

19. $562\frac{1}{2}$ per mille, $937\frac{1}{2}$ per mille.

20. A to B 37 miles, B to C 45 miles, C to A 52 miles.

21. 19 five-franc pieces, 11 two-franc pieces.

22. 10 bought, 5 burnt.

LXV.

1. 11, 1 ; 1, 11. 2. 25, 24. 3. 102, 98. 4. 4, 1.

5. 8, 10. 6. 1, 2 ; -50, 19. 7. 4, 1 ; $-\frac{1}{4}$, $-3\frac{1}{4}$.

8. 5, 2 ; $-3\frac{6}{7}$, $7\frac{19}{21}$. 9. 36, 38. 10. 25, 21.

11. $\frac{1}{3}$, $\frac{1}{4}$. 12. 4, 2. 13. $18\frac{1}{2}$, $15\frac{1}{2}$. 14. 2, 1.

15. 3, -5. 16. 7, 3. 17. 2, $\frac{1}{2}$; $\frac{3}{10}$, $-\frac{6}{5}$.

18. $\dfrac{a+\sqrt{a^2-4b}}{2}$, $\dfrac{a-\sqrt{a^2-4b}}{2}$

19. $\dfrac{\sqrt{a^2+4b}+a}{2}$, $\dfrac{\sqrt{a^2+4b}-a}{2}$.

20. $\frac{1}{2}b+\frac{1}{2}\sqrt{2a^2-b^2}$, $\frac{1}{2}b-\frac{1}{2}\sqrt{2a^2-b^2}$.

21. $\frac{1}{2}\sqrt{2a^2-b^2}+\frac{1}{2}b$, $\frac{1}{2}\sqrt{2a^2-b^2}-\frac{1}{2}b$.

22. $\dfrac{p+\sqrt{p^2-4abq}}{2a}$, $\dfrac{p-\sqrt{p^2-4abq}}{2b}$.

23. $\dfrac{\sqrt{p^2+4abq}+p}{2a}$, $\dfrac{\sqrt{p^2+4abq}-p}{2b}$.

LXVI.

1. 1, 2, 3. 2. 2, $3\frac{1}{2}$, $4\frac{1}{2}$. 3. 3, 4, 5. 4. 1, -2, 4.

5. 3, 4, 5, 6. 4, -5, 7. 7. -5, 3, -2.

8. $\dfrac{abc}{ab+ac-bc}$, &c.

9. $\sqrt{\dfrac{(c+a-b)(a+b-c)}{b+c-a}}$, &c.

10. $\dfrac{1}{\sqrt{bc}}$, $\dfrac{1}{\sqrt{ca}}$, $\dfrac{1}{\sqrt{ab}}$.

LXVII.

1. ± 3, ± 1; $\pm\frac{4}{3}\sqrt{6}$, $\pm\frac{1}{6}\sqrt{6}$.

2. ± 2, ± 3; ± 2, ± 1. 3. ± 10, ± 5.

4. 5, 4; 4, 5. 5. 9, 7; -7, -9.

6. ± 4, ∓ 3; $\pm\dfrac{5}{\sqrt{2}}$, $\pm\dfrac{5}{\sqrt{2}}$.

7. $\pm 5, \pm 2;\ \pm 2\sqrt{-1}, \mp 5\sqrt{-1}$.

8. 3, 2; 2, 3. 9. $\frac{1}{3}$, 3; -2, $-\frac{1}{2}$.

10. 10, 15; $-10\frac{7}{9}$, $-16\frac{1}{6}$. 11. 12, 3; -7, $-\frac{7}{4}$.

12. $\pm 10, \pm 3;\ \mp 14\sqrt{-\frac{1}{47}},\ \pm 57\sqrt{-\frac{1}{47}}$.

LXVIII.

1. 36. 2. 28, 32. 3. 16, 17. 4. $7\frac{1}{2}$, $3\frac{1}{2}$.

5. 88, 55. 6. 19, 10. 7. 10, 9.

8. 343 cubic in., 64 cubic in

9. $\frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(3+\sqrt{5}$.

10. $\frac{1}{2}(3+\sqrt{-3}), \frac{1}{2}(3-\sqrt{-3})$.

11. $\frac{1}{4}(5\pm\sqrt{5}), \pm\frac{1}{2}\sqrt{5})$.

12. 11, 7. 13. 73. 14. 32. 15. $\frac{5}{6}$, $-\frac{77}{88}$.

16. $\frac{17}{10}$. 17. 92. 18. 98. 19. 5 and 3.

20. £6000 at 7 per cent., £7000 at 6 per cent.

21. $\frac{5}{7}, \frac{-1.5}{0.5}$.

22. 5 years at 4 per cent., or 8 years at $2\frac{1}{2}$ per cent.

23. First cock in 8 hours, second cock in 6 hours.

LXIX.

3. $-\frac{\sqrt{2}+1}{2}$ and $\frac{\sqrt{2}-1}{2}$.

4. The line is bisected.

5. The line is bisected.

LXXI.

11. 2 : 3 : 4.

12. 2 : 3 : 4.

LXXIII.

1. 4. 2. 5 : 2. 3. 5.

4. $y^2=\frac{b^2}{a^2}(a^2-x^2)$. 5. $z=x+2x^2$.

6. 12. 8. 5 seconds.

9. $y=\frac{1}{a^2-b^2}\left\{(ap-bq)x+(aq-bp)\frac{ab}{x}\right\}$.

LXXIV.

1. $58\frac{1}{2}$ 2. 6. 3. -35. 4. $3\frac{3}{4}$.

5. $-1\frac{2}{21}$. 6. $\frac{n-1}{2}$.

7. $5\frac{7}{10}$, $5\frac{9}{10}$, $6\frac{1}{10}$, $6\frac{3}{10}$.

8. $d=5$, $n=9$. 9. 4, or 14. 10. 7.

11. 7, 11, 15, 19, 23. 12. £1 3s. 5½d., £326 11s. 3d.

13. 400 ft., 2704 ft.

14. $5\frac{1}{2}$, $8\frac{1}{2}$; or $-4\frac{3}{3}\frac{1}{4}$, $-8\frac{2}{3}\frac{9}{4}$.

15. 11, 35; or $-9\frac{5}{11}$, $-34\frac{6}{11}$.

16. $1+3+5+\ldots\ldots\ldots\ldots$

LXXV.

1. $2\frac{3}{4}$, $2\frac{2}{3}$. 2. $\frac{481}{1536}$, $\frac{8}{21}$. 3. $13\frac{4}{9}$, $13\frac{1}{2}$.

4. $\frac{1}{10}\cdot\frac{2^{\frac{5}{2}}+5^{\frac{5}{2}}}{2^{\frac{1}{2}}+5^{\frac{1}{2}}}$. 5. $\frac{2}{\sqrt{2}-1}$. 6. 6, 18, 54.

7. 1, $1\frac{1}{2}$, $2\frac{1}{4}$, $3\frac{3}{8}$, $5\frac{1}{16}$. 8. $2\frac{1}{2}$, 5, 10, 20, 40, 80, 160.

9. $\frac{5}{2}-\frac{1}{2}+\frac{1}{10}-\cdots$ 10. $a+\sqrt{a^2-b^2}$, $a-\sqrt{a^2-b^2}$.

11. $\frac{121}{148}$. 12. $\frac{4364}{12375}$.

LXXVI.

1. $3\frac{3}{11}$, $3\frac{3}{5}$. 2. $2\frac{2}{5}$, 3, 4, 6. 3. 8, 2.

LXXVII.

1. 3, -7, -2, -12.

2. -25, -14, $ab-1$.

3. $5a+6b=0$.

4. $a^2-b^2=0$.

LXXVIII.

1. 0, 6, 18. 2. 92, -168. 5. $c-b:a-c:b-a$.

6. $10:7:1$.

7. Combine with identity $(b-c)\,a+(c-a)\,b+(a-b)\,c=0$. 8. 2, 13. 9. 5, 5.

10. $\begin{vmatrix} a, & b, & -c \\ a', & b', & -c' \\ a'', & b'', & -c'' \end{vmatrix}=0.$

11. Assume each given quantity $=z$, and then eliminate x, y, z from the three equations.

Result. $\begin{vmatrix} a, & b, & -1 \\ a', & b', & -1 \\ a'', & b'', & -1 \end{vmatrix}=0.$

12. $ab'c+a'bc'=0$.

LXXIX.

1. -15, 8100, -972.

5. $-3:4:7:1$.

6. 4, 3, 2.

7. 1, 2, 3.

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